New realizations of the Richardson-Gaudin models in nuclear physics and condensed matter: The Hyperbolic model

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Thanks Franco for opened my eyes to the world algebraic models and dynamical symmetries when I was a PhD student in the late 70's.
Since then I have been always attracted by the concepts of exact solvability, symmetries, quantum integrability, quantum phase transitions , ….
This talk is a consequence of the development of these ideas along these years.

Happy 70th birthday

Integrals of motion of the Richardson-Gaudin Models

J. D., C. Esebbag and P. Schuck, Phys. Rev. Lett. 87, 066403 (2001).

- Pair realization of the SU(2) algebra

$$S_j^z = \frac{1}{2}\sum_m a_{jm}^+ a_{jm} - \frac{\Omega_j}{4}, \quad S_j^+ = \frac{1}{2}\sum_m a_{jm}^+ a^+_{j\bar{m}}$$

•Other possible realizations: Spin, two-level atoms (Lipkin), pairs with finite center of mass momentum (LOFF). Boson pair algebra SU(1,1).

• The most general combination of linear and quadratic generators, with the restriction of being hermitian and number conserving, is

$$R_i = S_i^z + 2g \sum_{j(\neq i)} \left[\frac{X_{ij}}{2} \left(S_i^+ S_j^- + S_i^- S_j^+ \right) + Y_{ij} S_i^z S_j^z \right]$$

$$for\ g = 0, \quad R_i = S_i^z$$

- The integrability condition $\left[R_i, R_j \right] = 0$ leads to

$$Y_{ij} X_{jk} + X_{jk} Y_{ki} + X_{ki} X_{ij} = 0$$

•These are the same conditions encountered by Gaudin (J. de Phys. 37 (1976) 1087) in a spin model known as the Gaudin magnet.

Gaudin (1976) found two independent solutions

Rational

$$X_{ij} = Y_{ij} = \frac{1}{\eta_i - \eta_j}$$

Hyperbolic

$$X_{ij} = \frac{1}{Sinh\left(x_i - x_j\right)} = 2\frac{\sqrt{\eta_i \eta_j}}{\eta_i - \eta_j}, \quad Z_{ij} = Coth\left(x_i - x_j\right) = \frac{\eta_i + \eta_j}{\eta_i - \eta_j}$$

Exact solution

$$R_i \left|\Psi\right\rangle = r_i \left|\Psi\right\rangle$$

Eigenstates of the Rational Model : Richardson Ansatz

$$\left|\Psi_{XXX}\right\rangle = \prod_{\alpha}\left(\sum_i \frac{1}{\eta_i - E_\alpha} S_i^+\right)\left|0\right\rangle, \quad \left|\Psi_{XXZ}\right\rangle = \prod_{\alpha}\left(\sum_i \frac{\sqrt{\eta_i}}{\eta_i - E_\alpha} S_i^+\right)\left|0\right\rangle$$

Any function of the *R* operators defines a valid integrable Hamiltonian. The Hamiltonian is diagonal in the basis of common eigenstates of the *R* operators.

•Within the pair representation two body Hamiltonians can be obtain by a linear combination of *R* operators:

$$H = \sum_l \varepsilon_l \, R_l \left(\eta, g\right) + C$$

•The parameters *g*, η´*s* and ε´*s* are arbitrary. There are 2 *M*+1 free parameters to define an integrable Hamiltonian in each of the families. (M number of single particle levels)

• The BCS Hamiltonian solved by Richardson can be obtained by from the *XXX* family by choosing $\eta = \varepsilon$. (Proof of integrability given by CRS)

$$H_{BCS} = \sum_i 2\varepsilon_i S_i^z + g \sum_{ij} S_i^+ S_j^-$$

•An important difference between RG models and other ES models is the large number of free parameters.

Some models derived from **rational** RG

- BCS Hamiltonian (Fermion and Boson).
- Generalized Pairing Hamiltonians (Fermion and Bosons).
- The Universal Hamiltonian of quantum dots.
- Central Spin Model.
- Generalized Gaudin magnets.
- Lipkin Model.
- Two-level boson models (IBM, molecular, etc..)
- Atom-molecule Hamiltonians (Feshbach resonances), or Generalized Jaynes-Cummings models,
- Breached superconductivity (Sarma state).
- Pairs with finite center of mass momentum, FFLO superconductivity.

Reviews: J.Dukelsky, S. Pittel and G. Sierra, Rev. Mod. Phys. 76, 643 (2004); G. Ortiz, R. Somma, J. Dukelsky y S. Rombouts. Nucl. Phys. B 7070 (2005) 401.

Exactly Solvable Pairing Hamiltonians

1) SU(2), Rank 1 algebra

$$H_R = \sum_i \varepsilon_i S_i^z - g \sum_{ij} S_i^+ S_j^-$$

2) SO(5), Rank 2 algebra, T=1 pairing.

$$H = \sum_i \varepsilon_i n_i - g \sum_{ij\tau} P_{i\tau}^+ P_{j\tau}$$

J. Dukelsky, V. G. Gueorguiev, P. Van Isacker, S. Dimitrova, B. Errea y S. Lerma H. PRL 96 (2006) 072503.

3) SO(6), Rank 3 algebra, color pairing

$$H = \sum_i \varepsilon_i n_i - g \sum_{ij\alpha} P_{i\alpha}^+ P_{j\alpha}$$

B. Errea, J. Dukelsky and G. Ortiz, PRA 79 05160 (2009)

4) SO(8), Rank 4 algebra, T=0,1 pairing.

$$H_{ST} = \sum_i \varepsilon_i n_i - g \sum_{ij\tau} P_{i\tau}^+ P_{j\tau} - g \sum_{ij\sigma} D_{i\sigma}^+ D_{j\sigma}$$

$$H_{3/2} = \sum_i \varepsilon_i n_i - g \sum_{ij} \left(P_{i00}^+ P_{j00} + \sum_{m=-2}^{2} P_{i2m}^+ P_{j2-m} \right)$$

S. Lerma H., B. Errea, J. Dukelsky and W. Satula. PRL 99, 032501 (2007).

The Hyperbolic Richardson-Gaudin Model

A particular RG realization of the hyperbolic family is the separable pairing Hamiltonian:

$$H = \sum_i \eta_i R_i = \sum_i \eta_i S_i^z - G \sum_{i,j} \sqrt{\eta_i \eta_j} S_i^+ S_j^-$$

With eigenstates:

$$\left|\Phi_M\right\rangle = \prod_{\alpha=1}^{M} \left(\sum_i \frac{\sqrt{\eta_i}}{\eta_i - E_\alpha} S_i^+ \right) \left|0\right\rangle, \quad E\left(\Phi_M\right) = \left\langle 0\right| H \left|0\right\rangle + \sum_{\alpha=1}^{M} E_\alpha$$

Richardson equations:

$$0 = \sum_i \frac{s_i}{\eta_i - E_\alpha} - \frac{Q}{E_\alpha} + \sum_{\alpha'(\neq\alpha)} \frac{1}{E_\alpha - E_{\alpha'}}, \quad 2Q = \frac{1}{G} - L + M - 2$$

The physics of the model is encoded in the exact solution. It does not depend on any particular representation of the Lie algebra

(p_x+ip_y) SU(2) spinless fermion representation

$$S_k^z = \frac{1}{2}\left(c_k^\dagger c_k + c_{-k}^\dagger c_{-k} - 1\right), \qquad S_k^+ = \frac{k_x + ik_y}{|k|} c_k^\dagger c_{-k}^\dagger = \left(S_k^-\right)^\dagger$$

Choosing $\eta_k = k^2$ we arrive to the p_x+ip_y Hamiltonian

$$H = \sum_{k(k_x>0)} \frac{k^2}{2}\left(c_k^\dagger c_k + c_{-k}^\dagger c_{-k}\right) - G \sum_{\substack{k,k' \\ (k_x, k_x' > 0)}} \left(k_x + ik_y\right)\left(k_x - ik_y\right) c_k^\dagger c_{-k}^\dagger c_{-k'} c_{k'}$$

M. I. Ibañez, J. Links, G. Sierra and S. Y. Zhao, Phys. Rev. B 79, 180501 (2009).
C. Dunning, M. I. Ibañez, J. Links, G. Sierra and S. Y. Zhao,, J. Stat. Mech. P080025 (2010).
S. Rombouts, J. Dukelsky and G. Ortiz, Phys. Rev. B. 82, 224510 (2010).

Why *p*-wave pairing?

- p_x+ip_y paired phase has been proposed to describe the A1 superfluid phase of ^3He.

- N. Read and D. Green (Phys. Rev. B 61, 10267 (2000)), studied the p_x+ip_y model. They showed that *p*-wave pairing has a QPT (2º order?) separating two gapped phases: a) a non-trivial topological phase. **Weak pairing**; b) a phase characterized by tightly bound pairs. **Strong pairing**.

- Moreover, there is a particular state in the phase diagram (the Moore-Read Pfafian) isomorphic to the $\nu=5/2$ fractional quantum Hall state.

- In polarized (single hyperfine state) cold atoms *p*-wave pairing is the most important scattering channel (*s*-wave is suppressed by Pauli). *p*-wave Feshbach resonances have been identified and studied. However, a *p*-wave atomic superfluid is unstable due to atom-molecule and molecule-molecule relaxation processes.

- Current efforts to overcome these difficulties. The great advantage is that the complete BCS-BEC transition could be explored.

From the exact solution

1) The Cooper pair wavefunction

$$\Gamma_\alpha^+ = \sum_k \frac{k_x + ik_y}{k^2 - E_\alpha} c_k^+ c_{-k}^+$$

- E_α real positive $\rightarrow$ Scattering state
- E_α complex $\rightarrow$ Cooper Resonance
- E_α real negative $\rightarrow$ Bound state

2) All pairons converge to zero (Moore-Read line)

$$G = \frac{1}{L - M + 1}, \qquad E_{GS} = 0$$

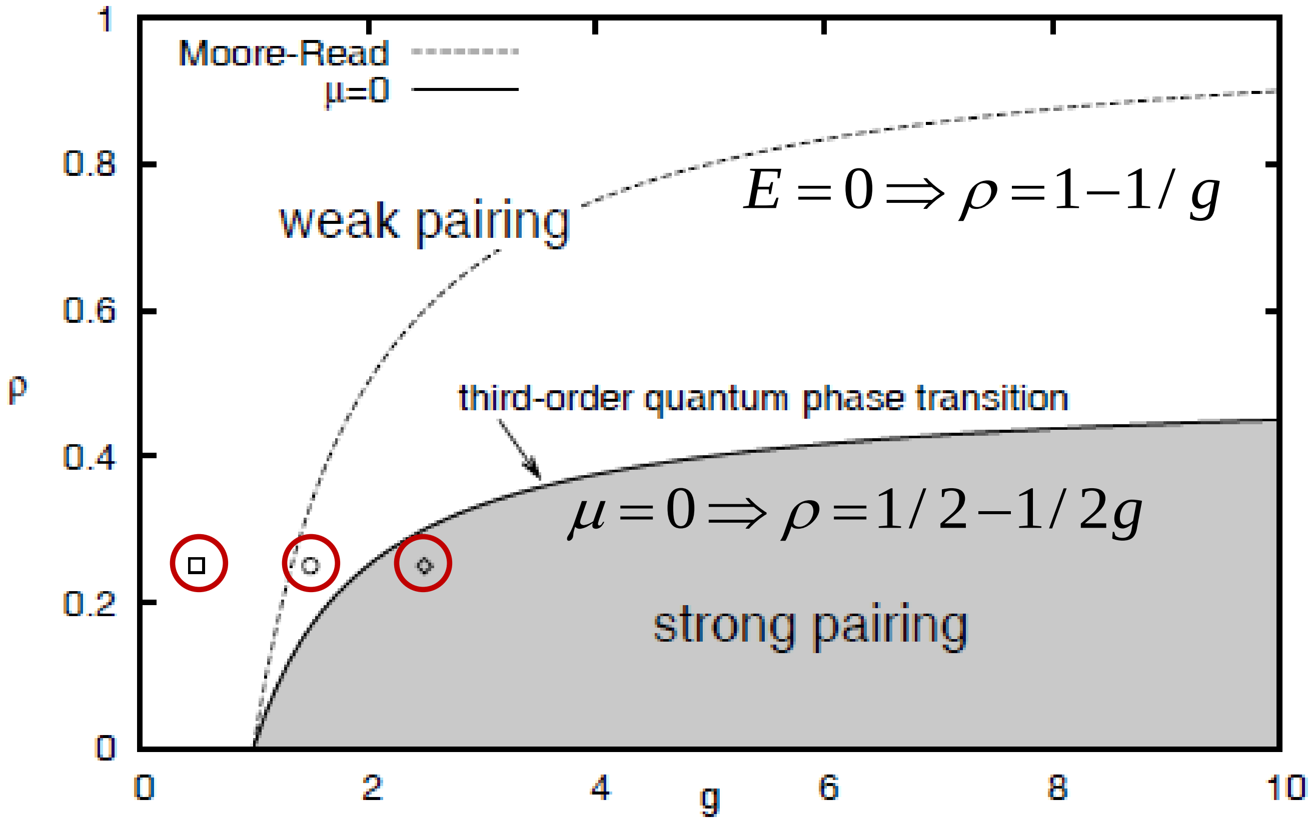
$$|\Phi\rangle_{Exact} = \left[\sum_{k, k_x > 0} \frac{1}{k_x - ik_y} c_k^\dagger c_{-k}^\dagger \right]^M |0\rangle = |PBCS\rangle$$

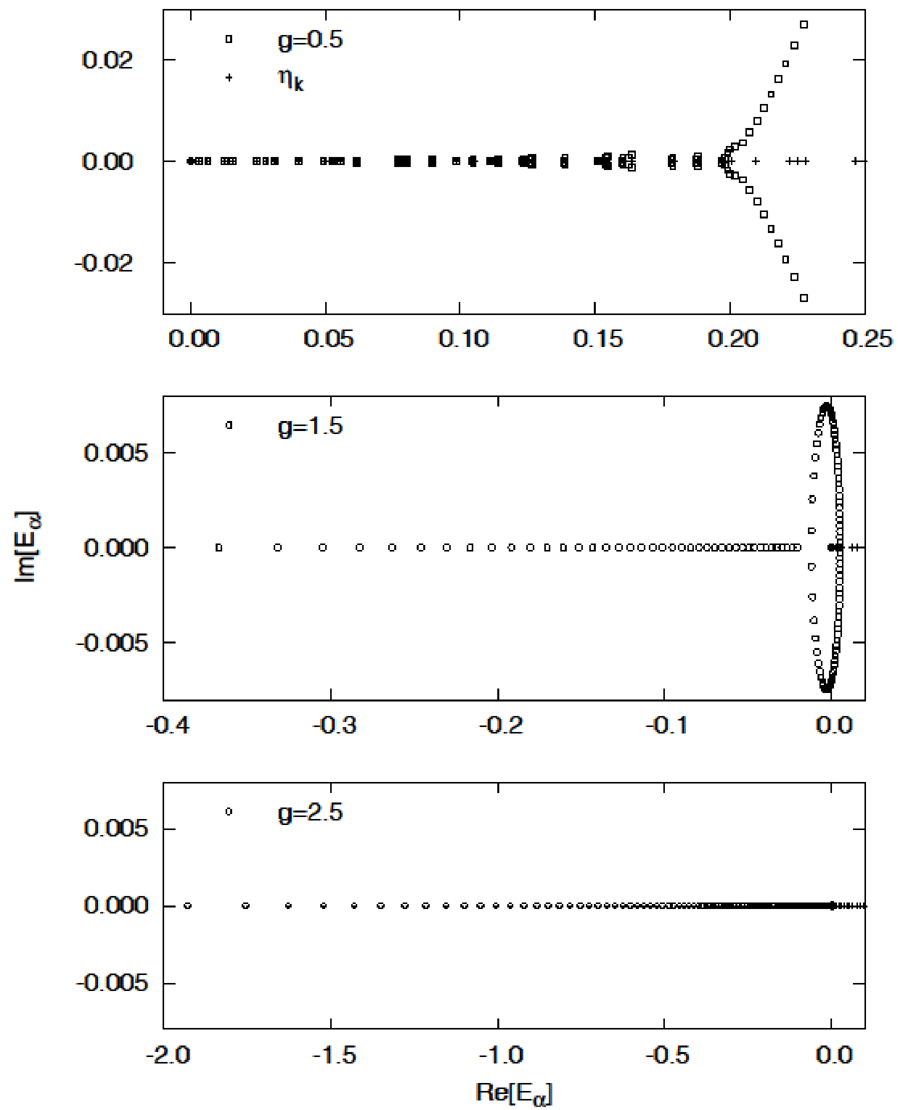
3) All pairons real and negative (Phase transition)

$$G \geq \frac{1}{L - 2M + 1}$$

Quantum phase diagram of the hyperbolic model

The phase diagram can be parametrized in terms of the density $\rho = M / L$
and the scaled coupling $g = GL$
In the thermodynamic limit the Richardson equations $\longrightarrow$ BCS equations





Exact solution in a 2D lattice with disk geometry of R=18 with total number of levels L=504 and M=126. (quarter filling)

$D \cong 10^{122}$

g=0.5 weak pairing

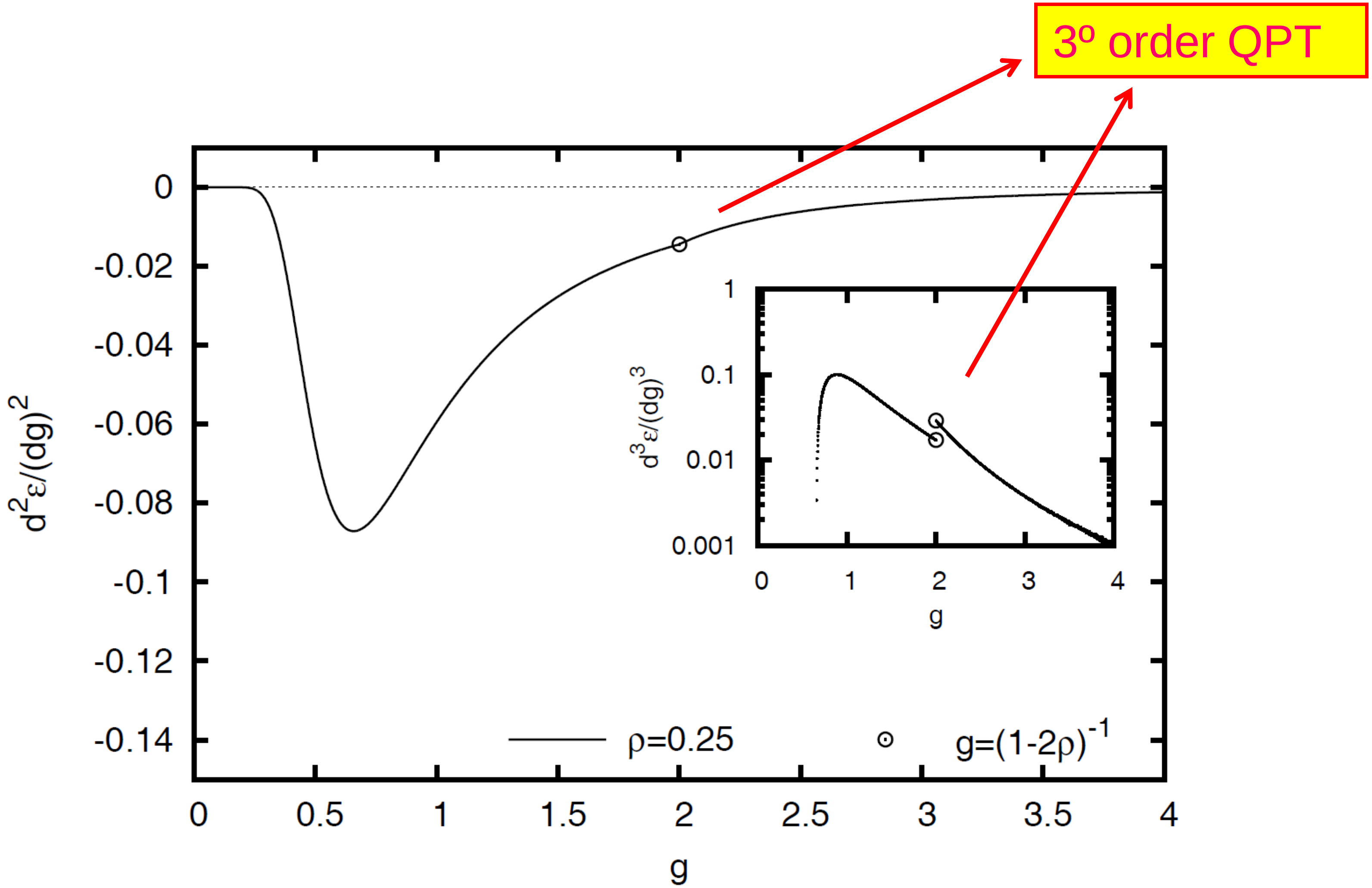
g=1.33 Moore-Read

g=1.5 weak pairing

g=2.0 QPT

g=2.5 strong pairing

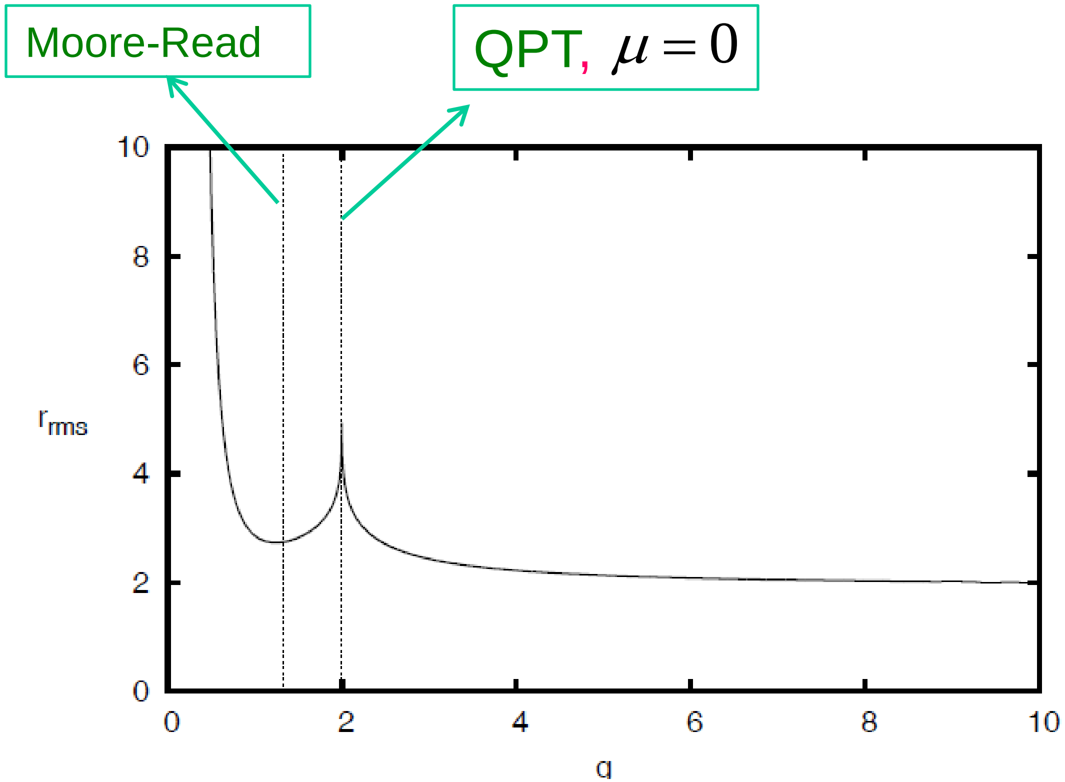
Higher order derivatives of the GS energy in the thermodynamic limit



Characterization of the QPT

In the thermodynamic limit the condensate wavefunction in k-space is:

$$\phi(k) = \langle \psi | c_k^\dagger c_{-k}^\dagger | \psi \rangle = u_k v_k$$



The length scale can be calculated as:

$$r_{rms}^2 = \frac{\int |\nabla \phi(k)|^2 dk}{\int |\phi(k)|^2 dk}$$

$$\lim_{\mu \to 0} \left(r_{rms}^2 \right) \approx Ln|\mu|$$

Accessible experimentally by quantum noise interferometry and time of flight analysis?

A similar analysis can be applied to the pairs in the exact solution

$$\Gamma_\alpha^+ = \sum_k \frac{k_x + ik_y}{k^2 - E_\alpha} c_k^+ c_{-k}^+$$

The root mean square $r^2_{rms,exact}$ of the pair wavefunction is finite for *E* complex or real and negative.

However, $r^2_{rms,exact} \Rightarrow \infty$ for $E\ real\ and \geq 0$

In strong pairing all pairs are bound $\left(E<0\right)$ and have finite radius.

At the QPT one pair energy becomes real an positive corresponding to a single deconfined Cooper pair on top of an ensemble of bound molecules.

The Hyperbolic Model in Nuclear Structure

J. Dukelsky, S. Lerma H., L. M. Robledo, R. Rodriguez-Guzman, S. Rombouts, PRC (in press)

The separable integrable Hyperbolic Hamiltonian

$$H = \sum_i \eta_i S_i^z - G \sum_{i,j} \sqrt{\eta_i \eta_j} S_i^+ S_j^-$$

Redefining the 0 of energy $\eta_i = 2(\varepsilon_i - \alpha)$, and absorbing the constant in the chemical potential μ

Exactly solvable H with non-constant matrix elements

$$H = \sum_i (\varepsilon_i - \mu) c_i^+ c_i - G \sum_{i,j} \sqrt{(\alpha - \varepsilon_i)(\alpha - \varepsilon_j)} c_i^+ c_{\bar{i}}^+ c_{\bar{j}} c_j$$

α is a new parameter that could serve as a cutoff.

Exact solution for a system with M=10 pairs and L=24 equally spaces levels

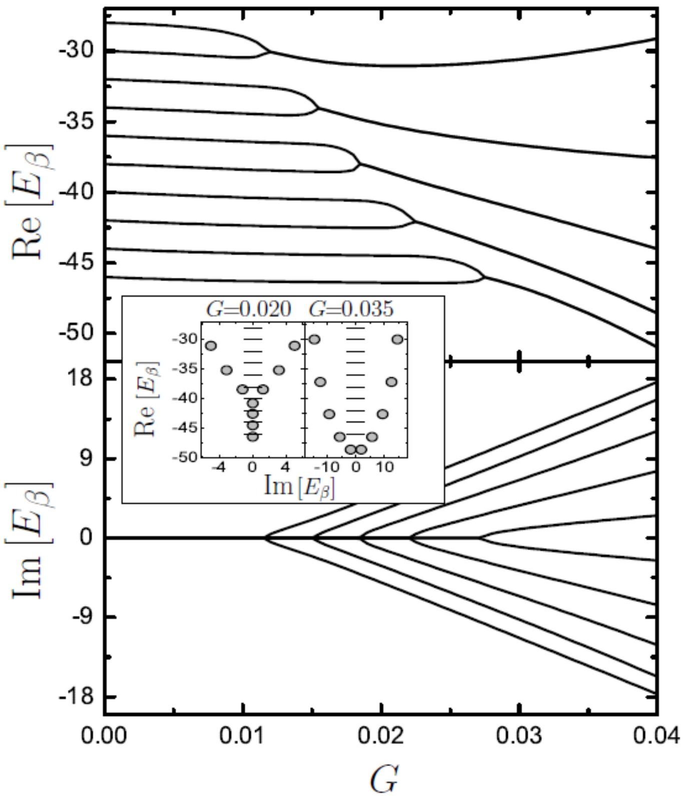
We choose α=24 as the energy cutoff.

For G →0 the 10 pair energies Eα converge to the lowest 10 η's.

Pair energies move down in energy till the two closest top the Fermi energy collapse into the η=-30 at G=-0.12 and then expand in the complex plane.

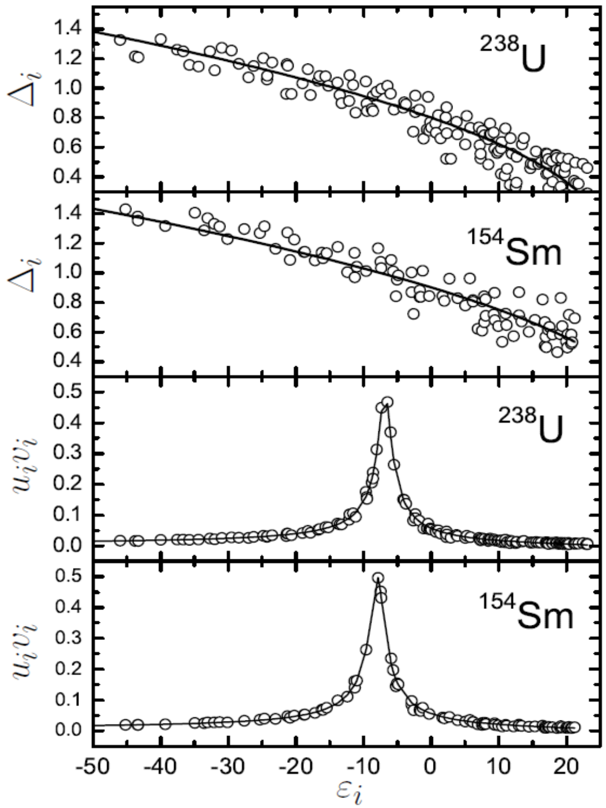
With increasing G values more pair energies collapse and expand in the complex plane.

Each complex E defines a correlated Cooper pair



Mapping of the Gogny force in the Canonical Basis

We fit the pairing strength *G* and the interaction cutoff α to the paring tensor $u_i v_i$ and the pairing gaps Δ_i of the Gogny HFB eigenstate in the Hartree-Fock basis.

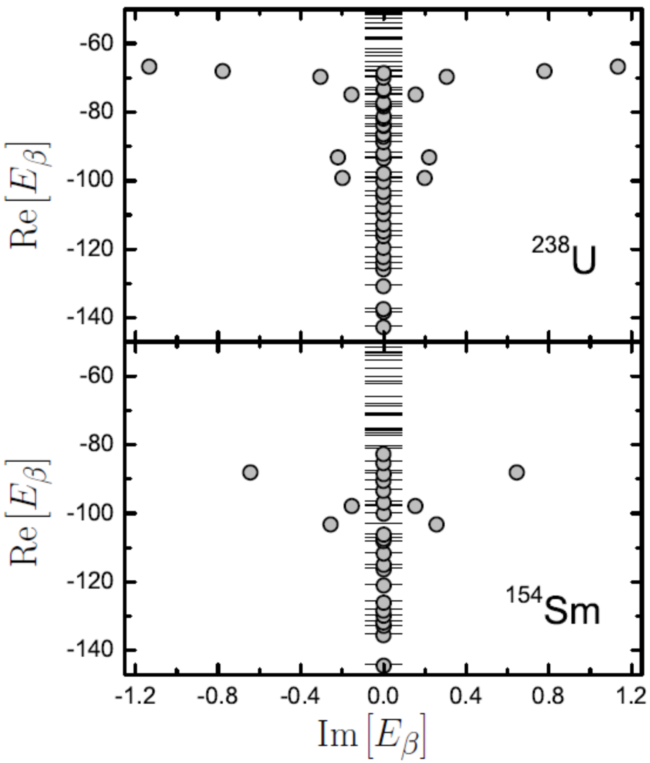


$$\Delta_i = G\sqrt{\alpha-\varepsilon_i}\sum_{i'}\sqrt{\alpha-\varepsilon_{i'}}\,u_{i'}v_{i'} = \Delta\sqrt{\alpha-\varepsilon_i}$$

$$u_i v_i = \frac{\Delta\sqrt{\alpha-\varepsilon_i}}{2\sqrt{(\varepsilon_i-\mu)^2+(\alpha-\varepsilon_i)\Delta^2}}$$

Protons
○ Gogny
_ Hyperbolico

	M	L	D	G	α	Δ	E^{G}_{corr}	E^{BCS}_{corr}	E^{Exa}_{corr}
^{154}Sm	31	95	9.9×10^{24}	2.2×10^{-3}	32.7	0.158	1.3254	1.0164	2.9247
^{238}U	46	148	4.8×10^{38}	2.0×10^{-3}	25.3	0.159	0.861	0.503	2.651



Summary

- Two new realizations of the Hyperpolic model in condensed matter and nuclear structure.
- From the analysis of the exact Richardson wavefunction we proposed a new view to the nature of the Cooper pairs in the BCS-BEC transition for p-wave pairing.
- The hyperbolic RG offers a unique tool to study a rare (topological?) 3º order QPT in the p_x+ip_y paired superfluid.
- We found that the root mean square size of the pair wave function diverges at the critical point. It could be a clear experimental signature of the QPT.
- We propose a new exactly solvable pairing Hamiltonian with two free parameters that reproduces the ground state properties of heavy nuclei as described by Gogny self-consistent mean field.
- It can be an excellent benchmark to test approximations beyond mean-field.
- It could be implemented in a self-consistent procedure of HF plus exact pairing to describe large nuclear regions of bound nuclei (work in progress).
- Extensions to the SO(5) Hyperbolic Richardson-Gaudin model to include T=1 p-n pairing for N~ Z nuclei are underway.