How to construct self/anti-self charge conjugate states for higher spins? (From neutrino to photon and all that)

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Outline. We construct self/anti-self charge conjugate (Majorana-like) states for the $(1/2, 0) \oplus (0, 1/2)$ representation of the Lorentz group, and their analogs for higher spins within the quantum field theory. The problem of the basis rotations and that of the selection of phases in the Dirac-like and Majorana-like field operators are considered. The discrete symmetries properties (P, C, T) are studied. The corresponding dynamical equations are presented. In the $(1/2, 0) \oplus (0, 1/2)$ representation they obey the Dirac-like equation with eight components, which has been first introduced by Markov. Thus, the Fock space for corresponding quantum fields is doubled (as shown by Ziino). The particular attention has been paid to the questions of chirality and helicity (two concepts which are frequently confused in the literature) for Dirac and Majorana states. We further review several experimental consequences which follow from the previous works of M.Kirchbach *et al.* on neutrinoless double beta decay, and G.J.Ni *et al.* on meson lifetimes.

I. MAJORANA SPINORS IN THE MOMENTUM REPRESENTATION.

During the 20th century various authors introduced *self/anti-self* charge-conjugate 4-spinors (including in the momentum representation), see, e. g., [Majorana, Bilenky, Ziino, Ahluwalia2]. Later [Lounesto, Dvoeglazov, Dvoeglazov2, Kirchbach, Rocha1] *etc* studied these spinors, they found corresponding dynamical equations, gauge transformations and other specific features of them. The definitions are:

$$C = e^{i\theta} \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \mathcal{K} = -e^{i\theta}\gamma^2\mathcal{K} \qquad (1)$$

is the anti-linear operator of charge conjugation. $\mathcal{K}$ is the complex conjugation operator. As usual, C transforms the $u-$ to $v-$ spinors, and *vice versa*. We define the *self/anti-self* charge-conjugate 4-spinors in the momentum space

$$C\lambda^{S,A}(\mathbf{p}) = \pm\lambda^{S,A}(\mathbf{p})\,, \qquad (2)$$

$$C\rho^{S,A}(\mathbf{p}) = \pm\rho^{S,A}(\mathbf{p})\,, \qquad (3)$$

where

$$\lambda^{S,A}(p^\mu) = \begin{pmatrix} \pm i\Theta\phi_L^*(\mathbf{p}) \\ \phi_L(\mathbf{p}) \end{pmatrix}\,, \qquad (4)$$

and

$$\rho^{S,A}(\mathbf{p}) = \begin{pmatrix} \phi_R(\mathbf{p}) \\ \mp i\Theta\phi_R^*(\mathbf{p}) \end{pmatrix}\,. \qquad (5)$$

The Wigner matrix is

$$\Theta_{[1/2]} = -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,, \qquad (6)$$

and ϕ_L, ϕ_R can be boosted with $\Lambda_{L,R} = (E + m \pm (\sigma\cdot\mathbf{p})/\sqrt{2m(E+m)}$ matrices.

Such definitions of 4-spinors differ, of course, from the original Majorana definition in x-representation:

$$\nu(x) = \frac{1}{\sqrt{2}}(\Psi_D(x) + \Psi_D^c(x)), \tag{7}$$

$C\nu(x) = \nu(x)$ that represents the positive real $C-$ parity field operator only. However, the momentum-space Majorana-like spinors open various possibilities for description of neutral particles (with experimental consequences, see [Kirchbach]). For instance, "for imaginary C parities, the neutrino mass can drop out from the single β decay trace and reappear in $0\nu\beta\beta$, a curious and in principle experimentally testable signature for a non-trivial impact of Majorana framework in experiments with polarized sources."

The rest λ and ρ spinors can be defined in accordance with (4,5) in analogious way with the Dirac spinors:

$$\lambda^S_\uparrow(\mathbf{0}) = \sqrt{\frac{m}{2}}\begin{pmatrix}0\\ i\\ 1\\ 0\end{pmatrix}, \lambda^S_\downarrow(\mathbf{0}) = \sqrt{\frac{m}{2}}\begin{pmatrix}-i\\ 0\\ 0\\ 1\end{pmatrix}, \tag{8}$$

$$\lambda^A_\uparrow(\mathbf{0}) = \sqrt{\frac{m}{2}}\begin{pmatrix}0\\ -i\\ 1\\ 0\end{pmatrix}, \lambda^A_\downarrow(\mathbf{0}) = \sqrt{\frac{m}{2}}\begin{pmatrix}i\\ 0\\ 0\\ 1\end{pmatrix}, \tag{9}$$

$$\rho^S_\uparrow(\mathbf{0}) = \sqrt{\frac{m}{2}}\begin{pmatrix}1\\0\\0\\-i\end{pmatrix}, \quad \rho^S_\downarrow(\mathbf{0}) = \sqrt{\frac{m}{2}}\begin{pmatrix}0\\1\\i\\0\end{pmatrix}, \tag{10}$$

$$\rho^A_\uparrow(\mathbf{0}) = \sqrt{\frac{m}{2}}\begin{pmatrix}1\\0\\0\\i\end{pmatrix}, \quad \rho^A_\downarrow(\mathbf{0}) = \sqrt{\frac{m}{2}}\begin{pmatrix}0\\1\\-i\\0\end{pmatrix}. \tag{11}$$

Thus, in this basis with the appropriate normalization ("mass dimension") the explicite forms of the 4-spinors of the second kind $\lambda^{S,A}_{\uparrow\downarrow}(\mathbf{p})$ and $\rho^{S,A}_{\uparrow\downarrow}(\mathbf{p})$ are:

$$\lambda^S_\uparrow(\mathbf{p}) = \frac{1}{2\sqrt{E+m}}\begin{pmatrix} ip_l \\ i(p^- + m) \\ p^- + m \\ -p_r \end{pmatrix}, \lambda^S_\downarrow(\mathbf{p}) = \frac{1}{2\sqrt{E+m}}\begin{pmatrix} -i(p^+ + m) \\ -ip_r \\ -p_l \\ (p^+ + m) \end{pmatrix} \quad (12)$$

$$\lambda^A_\uparrow(\mathbf{p}) = \frac{1}{2\sqrt{E+m}}\begin{pmatrix} -ip_l \\ -i(p^- + m) \\ (p^- + m) \\ -p_r \end{pmatrix}, \lambda^A_\downarrow(\mathbf{p}) = \frac{1}{2\sqrt{E+m}}\begin{pmatrix} i(p^+ + m) \\ ip_r \\ -p_l \\ (p^+ + m) \end{pmatrix} \quad (13)$$

$$\rho^S_\uparrow(\mathbf{p}) = \frac{1}{2\sqrt{E+m}}\begin{pmatrix} p^+ + m \\ p_r \\ ip_l \\ -i(p^+ + m)\end{pmatrix}, \rho^S_\downarrow(\mathbf{p}) = \frac{1}{2\sqrt{E+m}}\begin{pmatrix} p_l \\ (p^- + m) \\ i(p^- + m) \\ -ip_r\end{pmatrix} \tag{14}$$

$$\rho^A_\uparrow(\mathbf{p}) = \frac{1}{2\sqrt{E+m}}\begin{pmatrix} p^+ + m \\ p_r \\ -ip_l \\ i(p^+ + m)\end{pmatrix}, \rho^A_\downarrow(\mathbf{p}) = \frac{1}{2\sqrt{E+m}}\begin{pmatrix} p_l \\ (p^- + m) \\ -i(p^- + m) \\ ip_r\end{pmatrix}. \tag{15}$$

As claimed by [Ahluwalia2] λ and ρ 4-spinors are *not* the eigenspinors of the helicity. Moreover, λ and ρ are NOT the eigenspinors of the parity, as opposed to the Dirac case (in this representation $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} R$, where $R = (\mathbf{x} \to -\mathbf{x})$). The indices $\uparrow\downarrow$ should be referred to the chiral helicity quantum number introduced in the 60s, $\eta = -\gamma^5 h$. Ref. [SenGupta].

While

$$P u_\sigma(\mathbf{p}) = +u_\sigma(\mathbf{p})\,, P v_\sigma(\mathbf{p}) = -v_\sigma(\mathbf{p})\,, \tag{16}$$

we have

$$P \lambda^{S,A}(\mathbf{p}) = \rho^{A,S}(\mathbf{p})\,, P\rho^{S,A}(\mathbf{p}) = \lambda^{A,S}(\mathbf{p})\,, \tag{17}$$

for the Majorana-like momentum-space 4-spinors on the first quantization level. In this basis one has also the relations between the above-defined 4-spinors:

$$\rho^S_\uparrow(\mathbf{p}) = -i\lambda^A_\downarrow(\mathbf{p})\,, \rho^S_\downarrow(\mathbf{p}) = +i\lambda^A_\uparrow(\mathbf{p})\,, \tag{18}$$

$$\rho^A_\uparrow(\mathbf{p}) = +i\lambda^S_\downarrow(\mathbf{p})\,, \rho^A_\downarrow(\mathbf{p}) = -i\lambda^S_\uparrow(\mathbf{p})\,. \tag{19}$$

The normalizations of the spinors $\lambda^{S,A}_{\uparrow\downarrow}(\mathbf{p})$ and $\rho^{S,A}_{\uparrow\downarrow}(\mathbf{p})$ are the following ones:

$$\overline{\lambda}^S_\uparrow(\mathbf{p})\lambda^S_\downarrow(\mathbf{p}) = -im \quad , \quad \overline{\lambda}^S_\downarrow(\mathbf{p})\lambda^S_\uparrow(\mathbf{p}) = +im \quad , \tag{20}$$

$$\overline{\lambda}^A_\uparrow(\mathbf{p})\lambda^A_\downarrow(\mathbf{p}) = +im \quad , \quad \overline{\lambda}^A_\downarrow(\mathbf{p})\lambda^A_\uparrow(\mathbf{p}) = -im \quad , \tag{21}$$

$$\overline{\rho}^S_\uparrow(\mathbf{p})\rho^S_\downarrow(\mathbf{p}) = +im \quad , \quad \overline{\rho}^S_\downarrow(\mathbf{p})\rho^S_\uparrow(\mathbf{p}) = -im \quad , \tag{22}$$

$$\overline{\rho}^A_\uparrow(\mathbf{p})\rho^A_\downarrow(\mathbf{p}) = -im \quad , \quad \overline{\rho}^A_\downarrow(\mathbf{p})\rho^A_\uparrow(\mathbf{p}) = +im \quad . \tag{23}$$

All other conditions are equal to zero.

The dynamical coordinate-space equations are:

$$\begin{aligned} i\gamma^\mu\partial_\mu\lambda^S(x) - m\rho^A(x) &= 0\,, & (24)\\ i\gamma^\mu\partial_\mu\rho^A(x) - m\lambda^S(x) &= 0\,, & (25)\\ i\gamma^\mu\partial_\mu\lambda^A(x) + m\rho^S(x) &= 0\,, & (26)\\ i\gamma^\mu\partial_\mu\rho^S(x) + m\lambda^A(x) &= 0\,. & (27)\end{aligned}$$

These are NOT the Dirac equation. However, they can be written in the 8-component form as follows:

$$\begin{aligned} \left[i\Gamma^\mu\partial_\mu - m\right]\Psi_{(+)}(x) &= 0\,, & (28)\\ \left[i\Gamma^\mu\partial_\mu + m\right]\Psi_{(-)}(x) &= 0\,, & (29)\end{aligned}$$

with

$$\Psi_{(+)}(x) = \begin{pmatrix}\rho^A(x)\\ \lambda^S(x)\end{pmatrix}, \Psi_{(-)}(x) = \begin{pmatrix}\rho^S(x)\\ \lambda^A(x)\end{pmatrix}, \text{and } \Gamma^\mu = \begin{pmatrix}0 & \gamma^\mu\\ \gamma^\mu & 0\end{pmatrix}. \quad (30)$$

One can also re-write the equations into the two-component form. Thus, one obtains the [Feynman-Gell-Mann] equations. Similar formulations have been presented by M. [Markov], and by A. Barut and G. [Ziino]. The group-theoretical basis for such doubling has been given in the papers by Gelfand, Tsetlin and Sokolik [Gelfand], who first presented the theory in the 2-dimensional representation of the inversion group in 1956 (later called as "the Bargmann-Wightman-Wigner-type quantum field theory" in 1993).
The Lagrangian is

$$\mathcal{L} = \frac{i}{2}\,[\bar{\lambda}^S\gamma^\mu\partial_\mu\lambda^S - (\partial_\mu\bar{\lambda}^S)\gamma^\mu\lambda^S + \bar{\rho}^A\gamma^\mu\partial_\mu\rho^A - (\partial_\mu\bar{\rho}^A)\gamma^\mu\rho^A +$$
$$\bar{\lambda}^A\gamma^\mu\partial_\mu\lambda^A - (\partial_\mu\bar{\lambda}^A)\gamma^\mu\lambda^A + \bar{\rho}^S\gamma^\mu\partial_\mu\rho^S - (\partial_\mu\bar{\rho}^S)\gamma^\mu\rho^S] -$$
$$-m(\bar{\lambda}^S\rho^A + \bar{\rho}^A\lambda^S - \bar{\lambda}^A\rho^S - \bar{\rho}^S\lambda^A) \qquad (31)$$

The connection with the Dirac spinors has been found [Dvoeglazov, Kirchbach]. For instance,

$$\begin{pmatrix}\lambda^S_\uparrow(\mathbf{p})\\ \lambda^S_\downarrow(\mathbf{p})\\ \lambda^A_\uparrow(\mathbf{p})\\ \lambda^A_\downarrow(\mathbf{p})\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & i & -1 & i\\ -i & 1 & -i & -1\\ 1 & -i & -1 & -i\\ i & 1 & i & -1\end{pmatrix}\begin{pmatrix}u_{+1/2}(\mathbf{p})\\ u_{-1/2}(\mathbf{p})\\ v_{+1/2}(\mathbf{p})\\ v_{-1/2}(\mathbf{p})\end{pmatrix}. \qquad (32)$$

See also ref. [Gelfand, Ziino] and the discussion below. Thus, we can see that the two sets are connnected by the unitary transformations, and this represents itself the rotation of the spin-parity basis.
The sets of λ spinors and of ρ spinors are claimed to be *bi-orthonormal* sets each in the mathematical sense [Ahluwalia2], provided that overall phase factors of 2-spinors $\theta_1+\theta_2=0$ or π. For instance, on the classical level $\bar\lambda^S_\uparrow\lambda^S_\downarrow = 2iN^2\cos(\theta_1+\theta_2)$.

Several remarks have been given in the previous works:

- While in the massive case there are four λ-type spinors, two λ^S and two λ^A (the ρ spinors are connected by certain relations with the λ spinors for any spin case), in the massless case $\lambda^S_\uparrow$ and $\lambda^A_\uparrow$ may identically vanish, provided that one takes into account that $\phi_L^{\pm 1/2}$ may be the eigenspinors of $\sigma \cdot \hat{\mathbf{n}}$, the 2×2 helicity operator.

- It was noted that there exists the possibility of the generalization of the concept of the Fock space, which leads to the "doubling" Fock space [Gelfand, Ziino].

It was shown [Dvoeglazov] that the covariant derivative (and, hence, the interaction) can be introduced in this construct in the following way:

$$\partial_\mu \to \nabla_\mu = \partial_\mu - ig\text{Ł}^5 A_\mu \quad , \tag{33}$$

where $\text{Ł}^5 = \text{diag}(\gamma^5 \quad -\gamma^5)$, the 8×8 matrix. In other words, with respect to the transformations

$$\lambda'(x) \to (\cos\alpha - i\gamma^5 \sin\alpha)\lambda(x) \quad , \tag{34}$$
$$\overline{\lambda}'(x) \to \overline{\lambda}(x)(\cos\alpha - i\gamma^5 \sin\alpha) \quad , \tag{35}$$
$$\rho'(x) \to (\cos\alpha + i\gamma^5 \sin\alpha)\rho(x) \quad , \tag{36}$$
$$\overline{\rho}'(x) \to \overline{\rho}(x)(\cos\alpha + i\gamma^5 \sin\alpha) \tag{37}$$

the spinors retain their properties to be self/anti-self charge conjugate spinors and the proposed Lagrangian [Dvoeglazov, p.1472] remains to be invariant.

This tells us that while self/anti-self charge conjugate states have zero eigenvalues of the ordinary (scalar) charge operator but they can possess the axial charge (cf. with the discussion of [Ziino] and the old idea of R. E. Marshak – they claimed the same).
In fact, from this consideration one can recover the Feynman-Gell-Mann equation (and its charge-conjugate equation). It is re-written in the two-component form [Feynman-Gell-Mann]:

$$\left[\pi_\mu^- \pi^{\mu\,-} - m^2 - \frac{g}{2}\sigma^{\mu\nu} F_{\mu\nu}\right] \chi(x) = 0\,, \tag{38}$$

$$\left[\pi_\mu^+ \pi^{\mu\,+} - m^2 + \frac{g}{2}\widetilde{\sigma}^{\mu\nu} F_{\mu\nu}\right] \phi(x) = 0\,, \tag{39}$$

where already one has $\pi_\mu^\pm = i\partial_\mu \pm gA_\mu$, $\sigma^{0i} = -\widetilde{\sigma}^{0i} = i\sigma^i$, $\sigma^{ij} = \widetilde{\sigma}^{ij} = \epsilon_{ijk}\sigma^k$ and $\nu^{^{DL}}(x) = \text{column}(\chi \quad \phi)$.

Next, due to the transformations

$$\lambda'_S(\mathbf{p}) = \begin{pmatrix} \Xi & 0 \\ 0 & \Xi \end{pmatrix} \lambda_S(\mathbf{p}) \equiv \lambda^*_A(\mathbf{p}), \quad (40)$$

$$\lambda''_S(\mathbf{p}) = \begin{pmatrix} i\Xi & 0 \\ 0 & -i\Xi \end{pmatrix} \lambda_S(\mathbf{p}) \equiv -i\lambda^*_S(\mathbf{p}), \quad (41)$$

$$\lambda'''_S(\mathbf{p}) = \begin{pmatrix} 0 & i\Xi \\ i\Xi & 0 \end{pmatrix} \lambda_S(\mathbf{p}) \equiv i\gamma^0\lambda^*_A(\mathbf{p}), \quad (42)$$

$$\lambda^{IV}_S(\mathbf{p}) = \begin{pmatrix} 0 & \Xi \\ -\Xi & 0 \end{pmatrix} \lambda_S(\mathbf{p}) \equiv \gamma^0\lambda^*_S(\mathbf{p}) \quad (43)$$

with the 2×2 matrix Ξ defined as (ϕ is the azimuthal angle related with $\mathbf{p}$)

$$\Xi = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \quad , \quad \Xi\Lambda_{R,L}(\mathbf{p} \leftarrow \mathbf{0})\Xi^{-1} = \Lambda^*_{R,L}(\mathbf{p} \leftarrow \mathbf{0}) \ , \quad (44)$$

and corresponding transformations for λ^A, do *not* change the properties of bispinors to be in the self/anti-self charge-conjugate spaces, the

Majorana-like field operator ($b^\dagger \equiv a^\dagger$) admits additional phase (and, in general, normalization) transformations:

$$\nu^{ML\,\prime}(x^\mu) = [c_0 + i(\tau\cdot\mathbf{c})]\,\nu^{ML\,\dagger}(x^\mu)\,, \qquad (45)$$

where c_α are arbitrary parameters. The τ matrices are defined over the field of 2×2 matrices and the Hermitian conjugation operation is assumed to act on the c- numbers as the complex conjugation. One can parametrize $c_0 = \cos\phi$ and $\mathbf{c} = \mathbf{n}\sin\phi$ and, thus, define the $SU(2)$ group of phase transformations. One can select the Lagrangian which is composed from the both field operators (with λ spinors and ρ spinors) and which remains to be invariant with respect to this kind of transformations. The conclusion is: it is permitted the non-Abelian construct which is based on the spinors of the Lorentz group only (cf. with the old ideas of T. W. Kibble and R. Utiyama) . This is not surprising because the $SU(2)$ group is the sub-group of the extended Poincaré group (cf. [Ryder]).

The Dirac-like and Majorana-like field operators can be built from both $\lambda^{S,A}(\mathbf{p})$ and $\rho^{S,A}(\mathbf{p})$, or their combinations. For instance,

$$\Psi(x^\mu) \equiv \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2E_p}\sum_\eta \left[\lambda^S_\eta(\mathbf{p})\, a_\eta(\mathbf{p})\, \exp(-ip\cdot x) + \right.$$

$$+ \quad \left. \lambda^A_\eta(\mathbf{p})\, b^\dagger_\eta(\mathbf{p})\, \exp(+ip\cdot x)\right]. \tag{46}$$

The anticommutation relations are the following ones (due to the *bi-orthonormality*):

$$[a_{\eta'}(\mathbf{p}'), a^\dagger_\eta(\mathbf{p})]_\pm = (2\pi)^3 2E_p \delta(\mathbf{p}-\mathbf{p}')\delta_{\eta,-\eta'} \tag{47}$$

and

$$[b_{\eta'}(\mathbf{p}'), b^\dagger_\eta(\mathbf{p})]_\pm = (2\pi)^3 2E_p \delta(\mathbf{p}-\mathbf{p}')\delta_{\eta,-\eta'} \tag{48}$$

Other (anti)commutators are equal to zero: $([a_{\eta'}(\mathbf{p}'), b^\dagger_\eta(\mathbf{p})] = 0)$.

Finally, it is interesting to note that

$$\left[\nu^{ML}(x^\mu) + \mathcal{C}\nu^{ML\,\dagger}(x^\mu)\right]/2 = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2E_p}\sum_\eta \left[\begin{pmatrix} i\Theta\phi_L^{*\,\eta}(p^\mu) \\ 0 \end{pmatrix} a_\eta(p^\mu)e^{-ip\cdot x} + \right.$$

$$\left. + \begin{pmatrix} 0 \\ \phi_L^\eta(p^\mu) \end{pmatrix} a_\eta^\dagger(p^\mu)e^{ip\cdot x}\right], \qquad (49)$$

$$\left[\nu^{ML}(x^\mu) - \mathcal{C}\nu^{ML\,\dagger}(x^\mu)\right]/2 = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2E_p}\sum_\eta \left[\begin{pmatrix} 0 \\ \phi_L^\eta(p^\mu) \end{pmatrix} a_\eta(p^\mu)e^{-ip\cdot x} + \right.$$

$$\left. + \begin{pmatrix} -i\Theta\phi_L^{*\,\eta}(p^\mu) \\ 0 \end{pmatrix} a_\eta^\dagger(p^\mu)e^{ip\cdot x}\right], \qquad (50)$$

thus naturally leading to the Ziino-Barut scheme of massive chiral fields, ref. [Ziino].
The content of this Section is mainly based on the previous works of the 90s by D. V. Ahluwalia and by me (V. V. Dvoeglazov) dedicated to the Majorana-like momentum-representation 4-spinors. However, recently the interest to this model raised again [Rocha1, Rocha2].

II. CHIRALITY AND HELICITY.

- [Ahluwalia2] claimed "Incompatibility of Self-Charge Conjugation with Helicity Eignestates and Gauge Interactions". I showed that the gauge interactions of λ and ρ 4-spinors are different. As for the self/anti-self charge-conjugate states and their relations to helicity eigenstates the question is much more difficult, see below. Either we should accept that the rotations would have physical significance, or, due to some reasons, we should not apply the equivalence transformation to the discrete symmetry operators. As far as I understood [Ahluwalia2] paper, the latter standpoint is precisely his standpoint. He claimed [Ahluwalia2]: "Just as the operator of parity in the $(j,0)\oplus(0,j)$ representation space is independent of which wave equation is under study, similarly the operations of charge conjugation and time reversal do not depend on a specific wave equation. Within the context of the logical framework of the present paper, without this being true we would not even know how to define self-/anti self conjugate $(j,0)\oplus(0,j)$ spinors."

- Z.-Q. Shi and G. J. Ni promote a very extreme standpoint. Namely, "'the spin states, the helicity states and the chirality states of fermions in Relativistic Quantum Mechanics are entirely different: a spin state is helicity degenerate; a helicity state can be expanded as linear combination of the chirality states; the polarization of fermions in flight must be described by the helicity states" (see also his Conclusion Section [Shi]). In fact, they showed experimental consequences of their statement: "the lifetime of RH polarized fermions is always greater than of LH ones with the same speed in flight". However, we showed that the helicity, chiral helicity and chirality operators are connected by the unitary transformations. Do rotations have physical significance in their opinion?

- M. Markov wrote long ago [Markov] *two* Dirac equations with opposite signs at the mass term.

$$[i\gamma^\mu \partial_\mu - m]\,\Psi_1(x) = 0\,, \quad (51)$$
$$[i\gamma^\mu \partial_\mu + m]\,\Psi_2(x) = 0\,. \quad (52)$$

In fact, he studied all properties of this relativistic quantum model (while he did not know yet the quantum field theory in 1937). Next, he added and subtracted these equations. What did he obtain?

$$i\gamma^\mu \partial_\mu \chi(x) - m\eta(x) = 0\,, \quad (53)$$
$$i\gamma^\mu \partial_\mu \eta(x) - m\chi(x) = 0\,, \quad (54)$$

thus, χ and η solutions can be presented as some superpositions of the Dirac 4-spinors $u-$ and $v-$. These equations, of course, can be identified with the equations for λ and ρ we presented above. As he wrote himself he was expecting "new physics" from these equations.

- [SenGupta] and others claimed that the solutions of the equation (which follows from the general Sakurai method of derivation of relativistic quantum equations and it may describe both massive and massless $m_1 = \pm m_2$ states):

$$\left[i\gamma^\mu \partial_\mu - m_1 - m_2\gamma^5\right] \Psi = 0 \tag{55}$$

are *not* the eigenstates of chiral [helicity] operator $\gamma_0(\gamma \cdot \mathbf{p})/p$ in the massless limit. However, in the massive case the equation (55) has been obtained by the equivalence transformation of γ matrices.

- Barut and Ziino [Ziino] proposed yet another model. They considered γ^5 operator as the operator of charge-conjugation. Thus, the charge-conjugated Dirac equation has the different sign comparing with the ordinary formulation:

$$[i\gamma^\mu \partial_\mu + m]\Psi^c_{BZ} = 0\,, \tag{56}$$

and the so-defined charge conjugation applies to the whole system, fermions+electromagnetic field, $e \rightarrow -e$ in the covariant derivative. The concept of the doubling of the Fock space has been developed in Ziino works (cf. [Gelfand, Dvoeglazov5]). In their case, see above, their charge conjugate states are at the same time the eigenstates of the chirality.

Let us analize the above statements.

- The helicity operator is:

$$\hat{h} = \frac{1}{2}\begin{pmatrix} (\sigma\cdot\hat{\mathbf{p}}) & 0 \\ 0 & (\sigma\cdot\hat{\mathbf{p}}) \end{pmatrix} \tag{57}$$

However, we can do the equivalence transformation of the helicity $\hat{h}$-operator by the unitary matrix. It is known [Berg] that one can

$$\mathcal{U}_1(\sigma\cdot\mathbf{a})\mathcal{U}_1^{-1} = \sigma_3|\mathbf{a}|\,. \tag{58}$$

In the case of the momentum vector, one $(\mathbf{n}\equiv\hat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|)$

$$\mathcal{U}_1 = \begin{pmatrix} 1 & p_l/(p+p_3) \\ -p_r/(p+p_3) & 1 \end{pmatrix} \tag{59}$$

and

$$U_1 = \begin{pmatrix} \mathcal{U}_1 & 0 \\ 0 & \mathcal{U}_1 \end{pmatrix}\,. \tag{60}$$

Thus, we obtain:

$$U_1 \hat{h} U_1^{-1} = |\frac{\mathbf{n}}{2}| \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \quad (61)$$

Then, applying other unitary matrix U_3:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \\ = \gamma^5_{chiral} \,. \quad (62)$$

we transform to the basis, where helicity is equal (within the factor $\frac{1}{2}$) to γ^5, the chirality operator.

- [SenGupta] and others introduced the *chiral* helicity $\eta = -\gamma_5 h$, which is equal (within the sign and the factor $\frac{1}{2}$) to the well-known matrix α multiplied by $\mathbf{n}$. Again,

$$U_1(\alpha\cdot\mathbf{n})U_1^{-1} = |\mathbf{n}| \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} = \alpha_3|\mathbf{n}|\,. \tag{63}$$

with the same matrix U_1. And applying the second unitary transformation:

$$U_2\alpha_3 U_2^\dagger = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\end{pmatrix} \alpha_3 \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix} = \gamma^5_{chiral}\,, \tag{64}$$

we again come to the γ_5 matrix. The determinats are: $Det U_1 = 1 \neq 0$, $Det U_{2,3} = -1 \neq 0$. Thus, helicity, chirality and chiral helicity are connected by the unitary transformations.

- It is *not* surprising to have such a situation because the different helicity 2-spinors can be also connected *not only* by the anti-linear transformation [Ryder, Ahluwalia2] $\xi_h = (-1)^{1/2+h} e^{i\alpha_h} \Theta_{[1/2]} \mathcal{K} \xi_{-h}$, but the unitary transformation too. For example, when we parametrize the 2-spinors as in [Varshalovich, Dvoeglazov4]:

$$\xi_\uparrow = N\, e^{i\alpha} \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)\, e^{i\,\phi} \end{pmatrix}, \quad (65)$$

$$\xi_\downarrow = N\, e^{i\beta} \begin{pmatrix} \sin(\theta/2) \\ -\cos(\theta/2)\, e^{i\,\phi} \end{pmatrix}, \quad (66)$$

we obtain

$$\xi_\downarrow = U\xi_\uparrow = e^{i(\beta-\alpha)} \begin{pmatrix} 0 & e^{-i\phi} \\ -e^{i\phi} & 0 \end{pmatrix} \xi_\uparrow\,, \quad (67)$$

and

$$\xi_\uparrow = U^\dagger \xi_\downarrow = e^{i(\alpha-\beta)} \begin{pmatrix} 0 & -e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix} \xi_\downarrow\,. \quad (68)$$

To say that the 4-spinor is the eigenspinor of the *chiral helicity*, and, at the same time, it is *not!* the eigenspinor of the helicity operator (and that the physical results would depend on this) signifies the same as to say that rotations have physical significance on the fundamental level.

III. CHARGE CONJUGATION AND PARITY FOR $S=1$.

Several formalisms have been used for higher spin fields, e. g., [BargmannWigner, Weinberg]. The $2(2S+1)$ formalism gives the equations which are in some sense on an equal footing with the Dirac equation. For instance, for the spin-1 field the equation is

$$[\gamma_{\mu\nu} p_\mu p_\nu - m^2]\Psi(x) = 0\,, \qquad (69)$$

with the $\gamma_{\mu\nu}$ being the 6x6 covariantly-defined matrices. However, it was argued later that the signs before the mass terms should be opposite for charged particles of positive- and negative-frequencies [SankaranarayananGood, Ahluwalia1]:

$$[\gamma_{\mu\nu} p_\mu p_\nu - (\frac{i\partial/\partial t}{E}) m^2]\Psi(x) = 0\,. \qquad (70)$$

Hence, Ahluwalia *et al.* write: "The charge conjugation operation C must be carried through with a little greater care for bosons than for fermions within [this] framework because of $\wp_{u,v} = \pm 1$ factor in the mass term. For the $(1,0) \oplus (0,1)$ case, at the classical level we want

$$C: \left(\gamma_{\mu\nu} D_+^\mu D_+^\nu + m^2\right) u(x) = 0 \to \left(\gamma_{\mu\nu} D_-^\mu D_-^\nu - m^2\right) v(x) = 0, \tag{71}$$

where the local $U(1)$ gauge covariant derivatives are defined as: $D_+^\mu = \partial^\mu + i\, q\, A^\mu(x)\,,\quad D_-^\mu = \partial^\mu - i\, q\, A^\mu(x)$", Ref. [Ahluwalia1].
"These results read [Ref. [Ahluwalia2]]:

$$S_{[1]}^c = e^{i\vartheta_{[1]}^c} \begin{pmatrix} 0 & \Theta_{[1]} \\ -\Theta_{[1]} & 0 \end{pmatrix} \mathcal{K} \equiv C_{[1]} \mathcal{K}, \tag{72}$$

$$S_{[1]}^s = e^{i\vartheta_{[1]}^s} \begin{pmatrix} 0 & \mathbb{I}_3 \\ \mathbb{I}_3 & 0 \end{pmatrix} = e^{i\vartheta_{[1]}^s} \gamma_{00} \quad . \tag{73}$$

Note that neither $S_{[1/2]}^c$ nor $S_{[1]}^c$ are unitary (or even linear)." $\Theta_{[1]}$ is the 3x3 representation of the $\Theta_{[1/2]} = -i\sigma_2$.

"For spin-1 ... the requirement of self/anti-self charge conjugacy *cannot* be satisfied. That is, there does not exist a ζ [the phase factors between right- and left- 3-"spinors"] that can satisfy the spin-1 ... requirement"

$$S^c_{[1]}\,\lambda(p^\mu) = \pm\,\lambda(p^\mu)\;, \quad S^c_{[1]}\,\rho(p^\mu) = \pm\,\rho(p^\mu) \quad (?). \qquad (74)$$

This is due to the fact that $C^2_{[1]} = -\mathbb{I}$ within this definition of the charge conjugation operator.

"We find, however, that the requirement of self/anti-self conjugacy under charge conjugation can be replaced by the requirement of self/anti-self conjugacy under the operation of $\Gamma^5\, S^c_{[1]}$ [precisely, which was used by Weinberg in Ref. [Weinberg] due to the different choice of the equation for the negative-frequency 6-"bispinors"], where Γ^5 is the *chirality* operator for the $(1,\, 0) \oplus (0,\, 1)$ representation space and reads:
$\Gamma^5 = \begin{pmatrix} \mathbb{I}_3 & 0 \\ 0 & -\mathbb{I}_3 \end{pmatrix}$, with similar expressions for other spins.
The requirement

$$\left[\Gamma^5\, S^c_{[1]}\right]\, \lambda(p^\mu) = \pm\, \lambda(p^\mu)\ , \quad \left[\Gamma^5\, S^c_{[1]}\right]\, \rho(p^\mu) = \pm\, \rho(p^\mu) \qquad (75)$$

determines $\zeta^S_\lambda = +1 = \zeta^S_\rho$ for the self $\left[\Gamma^5\, S^c_{[1]}\right]$-conjugate "spinors" $\lambda^S(p^\mu)$ and $\rho^S(p^\mu)$; and $\zeta^A_\lambda = -1 = \zeta^A_\rho$ for the anti-self $\left[\Gamma^5\, S^c_{[1]}\right]$-conjugate "spinors" $\lambda^A(p^\mu)$ and $\rho^A(p^\mu)$".

The covariant equations for $\lambda-$ and $\rho-$ objects in the $(1,0) \oplus (0,1)$ representation have been obtained in Ref. [Dvoeglazov1]:

$$\gamma_{\mu\nu} p^\mu p^\nu \lambda^S_\uparrow(p^\mu) - m^2 \lambda^S_\downarrow(p^\mu) = 0, \gamma_{\mu\nu} p^\mu p^\nu \rho^S_\uparrow(p^\mu) - m^2 \rho^S_\downarrow(p^\mu) = 0, \tag{76}$$

$$\gamma_{\mu\nu} p^\mu p^\nu \lambda^S_\downarrow(p^\mu) - m^2 \lambda^S_\uparrow(p^\mu) = 0, \gamma_{\mu\nu} p^\mu p^\nu \rho^S_\downarrow(p^\mu) - m^2 \rho^S_\uparrow(p^\mu) = 0, \tag{77}$$

$$\gamma_{\mu\nu} p^\mu p^\nu \lambda^S_\rightarrow(p^\mu) + m^2 \lambda^S_\rightarrow(p^\mu) = 0, \gamma_{\mu\nu} p^\mu p^\nu \rho^S_\rightarrow(p^\mu) + m^2 \rho^S_\rightarrow(p^\mu) = 0, \tag{78}$$

$$\gamma_{\mu\nu} p^\mu p^\nu \lambda^A_\uparrow(p^\mu) + m^2 \lambda^A_\downarrow(p^\mu) = 0, \gamma_{\mu\nu} p^\mu p^\nu \rho^A_\uparrow(p^\mu) + m^2 \rho^A_\downarrow(p^\mu) = 0, \tag{79}$$

$$\gamma_{\mu\nu} p^\mu p^\nu \lambda^A_\downarrow(p^\mu) + m^2 \lambda^A_\uparrow(p^\mu) = 0, \gamma_{\mu\nu} p^\mu p^\nu \rho^A_\downarrow(p^\mu) + m^2 \rho^A_\uparrow(p^\mu) = 0, \tag{80}$$

$$\gamma_{\mu\nu} p^\mu p^\nu \lambda^A_\rightarrow(p^\mu) - m^2 \lambda^A_\rightarrow(p^\mu) = 0, \gamma_{\mu\nu} p^\mu p^\nu \rho^A_\rightarrow(p^\mu) - m^2 \rho^A_\rightarrow(p^\mu) = 0, \tag{81}$$

under the certain choice of the phase factors in the definition of left- and right- 3-objects.

On the quantum-field level we have to introduce the unitary operators for the charge conjugation and the parity in the Fock space:

$$U^c_{[S]}\Psi_{[S]}(x^\mu)(U^c_{[S]})^{-1} = C_{[S]}\Psi^\dagger_{[S]}(x^\mu), \tag{82}$$

$$U^s_{[S]}\Psi_{[S]}(x^\mu)(U^s_{[S]})^{-1} = \gamma^0\Psi_{[S]}({x'}^\mu). \tag{83}$$

For the spin $S=1/2$ they can be find in the well-known textbooks [ItsyksonZuber].

Next, let us return to the $(1/2,0)\oplus(0,1/2)$ representation. Roldao da Rocha *et al.* write [Rocha2]: "Now let one denotes the eigenspinors of the Dirac operator for particles and antiparticles respectively by $u_\pm(\mathbf{p})$ and $v_\pm(\mathbf{p})$. The subindex $\pm$ regards the eigenvalues of the helicity operator $(\sigma\cdot\widehat{\mathbf{p}})$. The parity operator acts as

$$Pu_\pm(\mathbf{p}) = +\,u_\pm(\mathbf{p}),\; Pv_\pm(\mathbf{p}) = -\,v_\pm(\mathbf{p}), \quad (84)$$

which implies that $P^2=\mathbb{I}$ in this case. The action of C on these spinors is given by

$$C(u_{\pm 1/2}(\mathbf{p})) = \mp v_\mp(\mathbf{p}), \qquad C(v_{\pm 1/2}(\mathbf{p})) = \pm u_{\mp 1/2}(\mathbf{p}). \quad (85)$$

which implies that $\{C,P\}=0$, [*anti*-commutator].
On the another hand the parity operator P acts on ELKO by

$$P\lambda^{\rm S}_{\mp,\pm}(\mathbf{p}) = \pm\, i\,\lambda^{\rm A}_{\pm,\mp}(\mathbf{p})\,, \qquad P\lambda^{\rm A}_{\mp,\pm}(\mathbf{p}) = \mp\, i\,\lambda^{\rm S}_{\pm,\mp}(\mathbf{p}), \quad (86)$$

and it follows that $[C,P]=0$ [when acting on the Majorana-like states]."
In the previous works of the 50s-60s, Ref. [NigamFoldy] it is this case

which has been attributed to the $Q=0$ eigenvalues (the truly neutral particles). You may compare these results with those of Refs. [Ahluwalia2, Dvoeglazov2, Dvoeglazov4], where the same statements have been done on the quantum-field level even at the earlier time comparing with [Rocha2]. The notation for 4-spinors used in the cited papers is a bit different. The acronym "ELKO" is (**almost**) the synonym for the self/anti-self charge conjugated states (the Majorana-like spinors). So, why the difference appeared in Eqs. (86) comparing with my previous results on the classical level?
In my papers, see, e.g., Ref. [Dvoeglazov1, Dvoeglazov2, Dvoeglazov4], I presented the explicite forms of the $\lambda-$ and $\rho-$ 2-spinors in the basis $\hat{S}_3\phi_{L,R}(\mathbf{0}) = \pm\frac{1}{2}\phi_{L,R}(\mathbf{0})$. The corresponding properties with respect to the parity (on the classical level) are different:

$$\gamma^0\lambda^S_\uparrow(p^{\mu'}) = +i\lambda^S_\downarrow(p^\mu),\, \gamma^0\lambda^S_\downarrow(p^{\mu'}) = -i\lambda^S_\uparrow(p^\mu), \tag{87}$$

$$\gamma^0\lambda^A_\uparrow(p^{\mu'}) = -i\lambda^A_\downarrow(p^\mu),\, \gamma^0\lambda^A_\downarrow(p^{\mu'}) = +i\lambda^A_\uparrow(p^\mu). \tag{88}$$

They have been presented in my previous works (and the corresponding

ones for $\rho-$ 4-spinors).
It is easy to find the correspondence between "the new notation", Refs. [Ahluwalia3, Rocha2] and the previous one. Namely, $\lambda^{S,A}_{\uparrow} \rightarrow \lambda^{S,A}_{-,+}$, $\lambda^{S,A}_{\downarrow} \rightarrow \lambda^{S,A}_{+,-}$. However, the difference is also in the choice of the basis for the 2-spinors (!). As in Ref. [Dvoeglazov3], Ahluwalia, Grumiller and da Rocha have chosen the well-known helicity basis (cf. [Varshalovich, Dvoeglazov4]). In my work of 2002 (published in 2004) I have shown that the helicity-basis 4-spinors satisfies the same Dirac equation, the parity matrix can be defined in the similar fashion as in the spinorial basis (according to the Itzykson-Zuber textbook procedure), but the helicity-basis 4-spinors are *not* the eigenspinors of the parity (in full accordance with the claims made in the 4th volume of the Landau course of theoretical physics and with the fact that $[\hat{h}, \hat{P}]_+ = 0$, Ref. [BLP]).

In this basis, the parity transformation ($\theta \to \pi - \theta$, $\phi \to \pi + \phi$) lead to the properties:

$$
\begin{aligned}
R\phi_L^-(\mathbf{0}) &= -ie^{i(\theta_2-\theta_1)}\phi_L^+(\mathbf{0}), & (89)\\
R\phi_L^+(\mathbf{0}) &= -ie^{i(\theta_1-\theta_2)}\phi_L^-(\mathbf{0}), & (90)\\
R\Theta(\phi_L^-(\mathbf{0}))^* &= -ie^{-2i\theta_2}\phi_L^-(\mathbf{0}), & (91)\\
R\Theta(\phi_L^+(\mathbf{0}))^* &= +ie^{-2i\theta_1}\phi_L^+(\mathbf{0}). & (92)
\end{aligned}
$$

This opposes to the spinorial basis, where, for instance: $R\phi_L^\pm(\mathbf{0}) = \phi_L^\pm(\mathbf{0})$. Further calculations are straightforward, and they indeed can lead to $[C, P]_- = 0$ when acting on the "ELKO" states, due to $[C, \gamma^5]_+ = 0$.

In the $(1,0) \oplus (0,1)$ representation the situation is similar. If we would like to extend the Nigam-Foldy conclusion, Ref. [NigamFoldy] (about $[C, P]_- = 0$ corresponds to the neutral particles even in the higher spin case (?)) then we should use the helicity basis on the classical level. However, on the level of the quantum-field theory (the "secondary" quantization) the situation is self-consistent. As shown in 1997, Ref. [Dvoeglazov2, Dvoeglazov4], we can obtain easily **both** cases (commutation and anti-commutation) on using $\lambda^{S,A}$ 4-spinors, which have been used earlier (in the basis *column*(1 0) *column*(0 1)).

IV. CONCLUSIONS.

We presented a review of the formalism for the momentum-space Majorana-like particles in the $(S,0)\oplus(0,S)$ representation of the Lorentz Group. The $\lambda-$ and $\rho-$ 4-spinors satisfy the 8- component analogue of the Dirac equation. Apart, they have different gauge transformations comparing with the usual Dirac 4-spinors. Their helicity, chirality and chiral helicity properties have been investigated in detail. These operators are connected by the given unitary transformations. At the same time, we showed that the Majorana-like 4-spinors can be obtained by the rotation of the spin-parity basis. Meanwhile, several authors have claimed that the physical results would be different on using calculations with these Majorana-like spinors. Thus, the $(S,0)\oplus(0,S)$ representation space (even in the case of $S=1/2$) has additional mathematical structures leading to deep physical consequences, which have not yet been explored before.

However, several claims made by other researchers concerning with chirality, helicity, chiral helicity should not be considered to be true until the time when experiments confirm them. Usually, it is considered that the rotations (unitary transformations) have no any physical consequences on the level of the Lorentz-covariant theories.

Next, we discussed the $[C, P]_\pm = 0$ dilemma for neutral and charged particles on using the analysis of the basis rotations and phases.

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