Quantum Mechanics

Dr. C. Chryssomalakos

Final Exam

1. (85 points) Consider a particle of mass m in one dimension, bounded by the potential

$$V(x) = -V_0 \delta(x) \tag{1}$$

with V_0 real and positive.

a) (15 points) Write down the time-independent Schrödinger equation for this potential. Solve it to obtain the normalized ground state wave function $\psi_0(x)$ and energy E_0. Are there any excited states?

b) (15 points) Consider now the motion of the particle in the potential

$$V(x) = -V_0 \left(\delta(x-a) + \delta(x+a)\right) \tag{2}$$

and let $\hat{H}$ be the corresponding Hamiltonian operator. Denote by $|\pm\rangle$ the states represented by $\psi_0(x \pm a)$ respectively (*notice:* $|+\rangle$ is centered at $-a$). Find the matrix elements of $\hat{H}$ in the basis $\{|+\rangle, |-\rangle\}$. Compute $\langle + \,|\, - \rangle$. Simplify your results assuming large separation of the two delta functions and keeping only first order corrections. What does *large* mean in this case?

c) (10 points) Find the eigenstates and eigenvalues of $\hat{H}$. What is the energy separation ΔE between the lowest two eigenstates?

d) (10 points) Assume the particle starts, at $t = 0$, in the state $|\alpha,\, t = 0\rangle = |+\rangle$. What is its state $|\alpha,\, t\rangle$ at time t? What is the probability to find the particle in the state $|-\rangle$ at time t? What is the probability to find it between 0 and a at time t?

e) (15 points) Compute $\langle \hat{x} \rangle_t \equiv \langle \alpha, t | \hat{x} | \alpha, t \rangle$, $\langle \hat{p} \rangle_t$, $\langle (\Delta \hat{x})^2 \rangle_t$ and $\langle (\Delta \hat{p})^2 \rangle_t$. Check that the Heisenberg uncertainty principle is satisfied.

f) (10 points) Find the eigenfunctions of $\hat{H}$ in the momentum basis. Express $|\alpha, t\rangle$ in the momentum basis. What is the probability that a measurement of the momentum of the particle at time t will return a value between 0 and p?

g) (10 points) Compute the correlation amplitude $C(t) \equiv \langle \alpha,\, t \,|\, \alpha,\, 0 \rangle$. What is the characteristic time ΔT needed in order for the initial wave function to change form appreciably? Compute the product $(\Delta T)(\Delta E)$ and comment.

2. (50 points) A beam of spin one-half particles with magnetic moment μ and velocity $\vec{v}_0$ enters a region of uniform magnetic field $\vec{B}$ along the positive z-axis, extending for a distance L along the direction of motion of the particles. At the moment of entry ($t = 0$) the magnetic moment of the particles is aligned with the positive x-axis.

a) (15 points) What is the spin state $|\text{out}\rangle$ of the particles when they exit the magnetic field? Express your answer in the basis $|S_z;\, \pm\rangle$.

b) (10 points) Find the unit vector $\hat{n}$ such that $|\text{out}\rangle = |\vec{S} \cdot \hat{n};\, +\rangle$. What should v_0 be in order for $\hat{n} = \hat{y}$? Is there more than one answer?

c) (15 points) Assuming a spread Δv around v_0 for the velocities of the particles in the beam, what is the angular spread $\Delta\theta$ of the vector $\hat{n}$ of part b)? What is the condition on Δv so that $\Delta\theta \ll 1$?

d) (10 points) Assume v_0 has the minimum of the values you found in part b), and the spread Δv is as in part c). The outgoing beam is passed through a Stern-Gerlach apparatus aligned with the y-axis. Estimate the percentage of particles deflected in the negative y-direction.

3. (85 points) Problem 25 from Chapter 2 of Sakurai (part (a) 35 points, part (b) 35 points, part (c) 15 points).

4. (105 points) A particle of mass m and charge q is constrained to move on the surface of a cylinder of radius a, whose axis is along $\hat{z}$. A magnetic field $\vec{B} = B_0 \frac{a}{\rho} \hat{\rho}$ is present ((ρ, ϕ, z) are cylindrical coordinates). The magnetic potential $\vec{A} = (0,\, -B_0 \frac{a}{\rho} z,\, 0)$ satisfies *curl* $\vec{A} = \vec{B}$ for $\rho \neq 0$ (do not worry about $\nabla \cdot \vec{B} = 0$ being violated at $\rho = 0$).

- **a) (15 points)** Write down the time-independent Schrödinger equation for the energy eigenfunction $\psi_E(\phi, z)$ (you will need to express the *kinematical* momentum operator in cylindrical coordinates and discard the radial part). Look for a solution in the form $\psi_E(\phi, z) = \Phi(\phi) Z(z)$. Show that a simple form for $\Phi(\phi)$, indexed by $m \in \mathbb{Z}$, indeed achieves the separation of variables.
- **b) (15 points)** Show that the resulting equation for $Z(z)$ describes a simple harmonic oscillator. Where is the quadratic potential centered, for each value of m? What is the characteristic frequency ω of the oscillator? Deduce the energy eigenvalues of the system. What is the ground state wavefunction for the particle? Is there any degeneracy?
- **c) (5 points)** Assume that, at $t = 0$, the particle has wavefunction

$$\psi(\phi, z, t = 0) = N\, e^{-z^2/2z_0^2}\, e^{i\phi}\,, \tag{3}$$

  where $z_0 \equiv \sqrt{\frac{\hbar}{m\omega}}$. Determine N so that ψ is normalized.
- **d) (15 points)** Compute $\psi(\phi, z, t)$.
- **e) (30 points)** Find $\psi(\phi, z, t)$ assuming the initial form

$$\psi(\phi, z, t = 0) = N'\, e^{-z^2/2z_0^2}\, (1 + e^{i\phi})\,. \tag{4}$$

- **f) (25 points)** Compute $\langle \hat{z} \rangle_t$, $\langle \hat{p}_z \rangle_t$, for the wavefunction of part e).

5. (50 points) A Stern-Gerlah apparatus is aligned with $\hat{z}$ and an electron beam is sent through it in the x-direction with velocity v. The Lorentz force $q\vec{v} \times \vec{B}$ is compensated by an horizontal (in the y-direction) electric field and will be ignored. There is left a vertical force F_μ, due to the interaction of the magnetic moment $\mu = e\hbar/2m_e c$ ($e < 0$) of the electrons with $\vec{B}$.

- **a) (5 points)** Find F_μ — justify carefully your answer.
- **b) (10 points)** Assume the beam has a width Δy (why can't we put $\Delta y = 0$?). Deduce the spread Δp_y in the y-momentum. Estimate the angular spread $\Delta\theta$ of the beam (assume non-relativistic speeds).
- **c) (20 points)** Show that the particles that pass through the apparatus a distance Δy off-center experience an horizontal component B_y of the magnetic field. Estimate the magnitude of this component in terms of the derivative $\partial B_x/\partial x$, evaluated on the axis of the apparatus, where $\vec{B}$ is along $\hat{z}$.
- **d) (15 points)** Find the force F_z due to the component of the field in part c). Assuming $\Delta\theta \ll 1$, what can you infer for the ratio F_y/F_μ? Comment on the feasibility of the experiment.

6. (30 points) Problem 14 from Chapter 2 of Sakurai (part (a) 15 points, part (b) 15 points).

7. (30 points) Problem 20 from Chapter 2 of Sakurai (part (a) 15 points, part (b) 15 points).

8. (30 points) Problem 27 from Chapter 2 of Sakurai.