Quantum Mechanics

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Second Midterm Exam

Notice: in the following, $\hbar = c = 1$.

1. (70 points) A particle is scattered by a spherically symmetric potential,

$$V(\vec{r}) = V(r) = V_0 e^{-\mu r} \sin \tilde{k} r \cos 3\tilde{k} r \,. \tag{1}$$

Assume the incoming wavefunction to be a plane wave travelling along the z-axis,

$$\psi(\vec{r}) = \frac{e^{ikz}}{(2\pi)^{3/2}} \,. \tag{2}$$

a) (10 points) Using the first order Born approximation, find the scattering amplitude $f^{(1)}(\theta)$. Leave your answer in terms of a radial integral but do perform all angular integrations.

b) (30 points) The Fourier transform of a function $V(\vec{r})$ is, in general, defined as

$$\tilde{V}(\vec{Q}) = \int_{\text{all space}} d^3\vec{r}\, e^{i\vec{Q}\cdot\vec{r}} V(\vec{r}) \,. \tag{3}$$

For the potential of (1), a plot of $\tilde{V}(\vec{Q}) = \tilde{V}(Q)$ (for $\tilde{k} = 20$, $\mu = 1$ and $4\pi V_0 = 1$) is given in the figure on page 3.

1. Give a rough sketch of $|f(\theta)|$ in polar coordinates, for $k = \tilde{k}$.
2. Compute approximately the values of θ for which $f(\theta) = 0$.
3. Compute approximately the maximum value of $f(\theta)$.

c) (30 points) Repeat part (b) for $k = 2\tilde{k}$.

2. (50 points) Consider scattering by a hard sphere of radius R,

$$V(r) = \begin{cases} \infty & \text{for } r \leq R \\ 0 & \text{for } r > R \end{cases} \,. \tag{4}$$

a) (15 points) Derive the transcendental equation that determines the phase-shifts δ_l. Assuming $x \equiv kR \ll 1$, find an approximate expression for δ_l, $l = 0, 1$, to lowest non-trivial order in x.

b) (20 points) Denote by $\sigma_{\text{tot}}^{(l)}$ the contribution to the total scattering cross-section of the l-partial wave. With the assumptions of part (a), what is the dependence of $\sigma_{\text{tot}}^{(l)}$, $l = 0, 1$, on the energy E of the incoming particle? For $x = 0.01$, what is the ratio $\sigma_{\text{tot}}^{(1)}/\sigma_{\text{tot}}^{(0)}$?

c) (15 points) Find an approximate expression for the scattering amplitude $f(\theta)$, considering only the contributions of the $l = 0, 1$ partial waves.

The spherical bessel functions for $l = 0, 1$ are

$$j_0(\rho) = \frac{\sin\rho}{\rho}\,, \qquad n_0(\rho) = -\frac{\cos\rho}{\rho}\,, \qquad j_1(\rho) = \frac{\sin\rho}{\rho^2} - \frac{\cos\rho}{\rho}\,, \qquad n_1(\rho) = -\frac{\cos\rho}{\rho^2} - \frac{\sin\rho}{\rho}\,. \tag{5}$$

3. (60 points) Given a Dirac bispinor,

$$\psi(x) = \begin{pmatrix} 1 \\ 0 \\ \alpha \\ 0 \end{pmatrix} e^{-i(Et - pz)} \,. \tag{6}$$

a) (20 points) Find under what conditions is the above $\psi(x)$ a solution of the Dirac equation,

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0 \,. \tag{7}$$

b) (25 points) An inertial frame S' moves along the z-axis with rapidity β. Find the Lorentz-transformed bispinor $\psi'(x')$ observed in S'. Is there any value of β for which $\alpha = 0$?

c) (15 points) What is the probability that a particle described by $\psi(x)$ may be found inside a sphere of radius R centered at the origin? What is the probability to be observed as an antiparticle? What is the limiting value of this latter probability when $p \to \infty$?

4. (40 points) Five identical noninteracting spin-1/2 particles of mass m are in a "spherical" (2D) harmonic oscillator potential of frequency ω.

a) (15 points) What are the four lowest eigenstates of the system, taking into account statistics? Is there any degeneracy?

b) (10 points) What would be the effect to the spectrum of the system of a small Coulombic interaction among the particles?

c) (15 points) Repeat parts (a) and (b) for five identical spin-1 particles.