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Normal Coordinates and Primitive Elements in the Hopf Algebra of Renormalization

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Abstract: We introduce normal coordinates on the infinite dimensional group G introduced by Connes and Kreimer in their analysis of the Hopf algebra of rooted trees. We study the primitive elements of the algebra and show that they are generated by a simple application of the inverse Poincaré lemma, given a closed left invariant 1-form on G. For the special case of the ladder primitives, we find a second description that relates them to the Hopf algebra of functionals on power series with the usual product. Either approach shows that the ladder primitives are given by the Schur polynomials. The relevance of the lower central series of the dual Lie algebra in the process of renormalization is also discussed, leading to a natural concept of k-primitiveness, which is shown to be equivalent to the one already in the literature.

Contents

1. Introduction

The process of renormalization in quantum field theory has been substantially elucidated in recent years. In a series of papers (see, e.g., [11, 7, 2, 9] and references therein), a Hopf algebra structure has been identified that greatly simplifies its combinatorics. This, in turn, has led to the development of an underlying geometric picture, involving an infinite dimensional group manifold G, the coordinates of which are in one-to-one correspondence with (classes of) 1PI superficially divergent Feynman diagrams of the theory. The latter are indexed by a type of graphs known as (decorated) rooted trees, which capture the subdivergence structure of the diagram. The forest formula prescription for the renormalization of a diagram then is translated into a series of operations on the corresponding rooted tree and the latter have been shown to deliver standard Hopf algebraic quantities, like the coproduct and the antipode of the rooted tree. The above results were obtained using a powerful mixture of algebraic and combinatoric techniques that brought to light unexpected interconnections with noncommutative geometry, among several other fields.

The complexity of the full Hopf algebra of decorated rooted trees is, in many respects, overwhelming. Even in the simplest cases, one is confronted with an infinite set of available decorations for the vertices of the rooted trees, originating in the infinite number of primitive divergent diagrams appearing in the underlying theory. It is rather fortunate then that the considerably simpler algebra of rooted trees with a single decoration seems to capture many of the features of realistic theories. It is for this reason that it has been studied extensively, as a first step towards an understanding of the full theory. Of primary importance, given their rôle in renormalization theory, is the study of the primitive elements of the above Hopf algebra. These correspond to sums of products of diagrams with the property that their renormalization involves a single subtraction. In Ref. [3], an ansatz is presented for a (conjectured) infinite family of such elements, corresponding to the ladder generators of the algebra, i.e., to trees whose every vertex has fertility at most one. Furthermore, dealing with the general case, a set of vertex-increasing operators is constructed that generates new primitive elements from known ones. As the number of primitive elements increases rapidly with increasing number of vertices, this approach necessitates the introduction of new operators in each step, a task that has not yet been systematized.

Our motivation in this paper is two-fold. On a general, methodological level, we argue that the above algebraic/combinatoric approach, with all its multiple successes, should nevertheless be complemented by a differential geometric one, which, we feel, has not been sufficiently considered in the literature. On a second, more concrete level, we provide support for our claim, by showing how a simple application of the inverse Poincaré lemma reduces the search for primitive elements to that of closed, left invariant (LI) 1-forms on G. For the case of the ladder primitives, we give a simple generating formula that identifies them with the Schur polynomials. Our discussion uses the normal coordinates on the group, a choice that leads naturally to a concept of k-primitiveness, associated with the lower central series of the dual Lie algebra – we prove that this coincides with the k-primitiveness introduced in Ref. [3]. We discuss the rôle of the new coordinates in renormalization, using the toy model realization of Ref. [10], while also commenting on similar results obtained for the more realistic heavy quark model of [2].

2. Differential Geometry á la Hopf

We will be dealing with differential geometric concepts expressed in Hopf algebraic terms. We opt for this formulation having in mind the transcription of our results for the non-commutative case – Hopf algebras are ideally suited to this task. We start by providing a short dictionary between the two languages and establish the notation, assuming nevertheless familiarity with the basic definitions.

Two algebras will be of main interest to us: on the one hand we have the (commutative, non-cocommutative) algebra $\mathcal{A}$ of functions on a (possibly infinite dimensional) group manifold, generated by $\{\phi^A\}$, with A ranging in an index set – we denote by a, $b, \ldots$ general elements of $\mathcal{A}$. On the other hand, we have the (non-commutative, cocommutative) universal enveloping algebra $\mathcal{U}$ of the Lie algebra of the group. We actually work with a suitable completion of $\mathcal{U}$, so as to allow exponentials of its generators Z_A, which we identify with the points of the manifold[1] – we denote by $x, y, \ldots$ general elements of $\mathcal{U}$ (we use $g, g', \ldots$ if we refer to group elements in particular).

Both algebras are Hopf algebras. For $\mathcal{A}$, the *coproduct* $\Delta(a) \equiv a_{(1)} \otimes a_{(2)}$ codifies left and right translations

$$L_g^*(a)(\cdot) = a_{(1)}(g)a_{(2)}(\cdot), \tag{1}$$

and similarly for right translations. For $\mathcal{U}$, it expresses Leibniz's rule, $\Delta(Z) = Z \otimes 1 + 1 \otimes Z$, for the left-invariant generator Z. The two Hopf algebras are *dual* , via the *inner product* (also called *pairing*)

$$\langle \cdot , \cdot \rangle : \mathcal{U} \otimes \mathcal{A} \to \mathbb{C}, \qquad x \otimes a \mapsto \langle x, a \rangle \,, \tag{2}$$

which, when x stands for a generator Z, amounts to taking the derivative of a along x and evaluating it at the identity. For $x = g$, the above definition produces a Taylor series expansion of a at the identity which gives, for a analytic, the value $a(g)$ of a at the point g. The coproduct in $\mathcal{A}$ is dual to the product in $\mathcal{U}$ via

$$\langle xy, a \rangle = \left\langle x \otimes y, a_{(1)} \otimes a_{(2)} \right\rangle \tag{3}$$

and *vice-versa*. We usually work with *dual bases*, so that Z_A only gives 1 when paired with ϕ^A, while its inner product with all other ϕ's, as well as with all products of ϕ's, vanishes. Given a Poincaré–Birkhoff–Witt basis $\{f^i\}$ for $\mathcal{A}$,

$$\{f^i\} = \{1, \phi^A, \phi^A\phi^B, \ldots\}, \tag{4}$$

one can build a dual basis $\{e_i\}$ for the entire $\mathcal{U}$ by adjoining to the above Z's polynomials in them, $\{e_i\} = \{1, Z_A$, quadratic, cubic, $\ldots\}$, with $\left\langle e_i, f^j \right\rangle = \delta_i^j$ – this, in general, involves a non-trivial calculation.

To every element a of $\mathcal{A}$ we can associate a LI 1-form Π_a, given by

$$\Pi_a = S(a_{(1)})da_{(2)}, \tag{5}$$

[1] The particular group we deal with in Sect. 3 is non-compact and infinite dimensional. Nevertheless, in this paper, we only consider elements that correspond to exponentials of linear combinations of the generators. For a readable account of what we might be missing in doing so, see Ref. [12].

d being the exterior derivative and S the antipode in $\mathcal{A}$. Π is linear, while on products it gives

$$\Pi_{(ab)} = \Pi_a \epsilon(b) + \epsilon(a)\Pi_b, \qquad \mathcal{A} \text{ commutative}, \tag{6}$$

where ϵ is the counit in $\mathcal{A}$. We take all generators ϕ^T of $\mathcal{A}$ to be counitless, i.e., we choose functions that vanish at the identity of the group, except for the unit function $1_{\mathcal{A}}$ (which we often write as just 1). This implies that Π only returns a non-zero result when applied to the generators and vanishes on all products, as well as on $1_{\mathcal{A}}$. The Maurer-Cartan (MC) equations take the form

$$d\Pi_a = -\Pi_{a_{(1)}} \Pi_{a_{(2)}}. \tag{7}$$

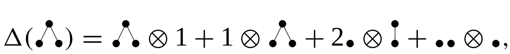
Using (6), one sees that only the bilinear part of the coproduct contributes to the MC equations.

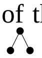
3. The Hopf Algebra of Rooted Trees and Its Dual

3.1. Functions. We specialize the general considerations of the previous section to the Connes–Kreimer algebra of renormalization. For a detailed exposition we refer the reader to [10, 6, 8] and references therein, we give here only a brief account of the basic definitions and some illustrative examples. $\mathcal{A}$ is now the Hopf algebra $\mathcal{H}_R$ of functions generated by ϕ^T, where T is a rooted tree. This means that the group manifold G is, in this case, infinite dimensional, with one dimension for every rooted tree – the ϕ's are coordinate functions on this manifold. The group law is encoded in the coproduct

$$\Delta(\phi^T) = \phi^T \otimes 1 + 1 \otimes \phi^T + \sum_{\text{cuts } C} \phi^{P^C(T)} \otimes \phi^{R^C(T)}. \tag{8}$$

The sum in the above definition is over *admissible* cuts, i.e., cuts that may involve more than one edge (*simple* cuts) but such that there is no more than one simple cut on any path from the root downwards. $R^C(T)$ is the part that is left containing the root of T while $P^C(T)$ is the product of all branches cut, e.g.


(9)

where we let a tree T itself denote the corresponding function ϕ^T, a convention freely used in the rest of the paper. The factor 2 on the r.h.s. appears because there are two possible cuts on  generating the corresponding term. A convenient way to recast (8) as a single sum, is to introduce a *full* and an *empty cut*, above and below any tree T respectively, e.g.,

(10)

We rewrite (8) in the form

$$\Delta(\phi^T) = \sum_{\text{cuts } C'} \phi^{P^{C'}(T)} \otimes \phi^{R^{C'}(T)}, \tag{11}$$

where the above two extra cuts, included in C', produce the primitive part of the coproduct. Notice that Δ respects the grading given by the number $v(T)$ of vertices of a tree T. We call this the v-degree of ϕ^T, denote it by $\deg_v(\phi^T)$, and extend it to monomials as the sum of the v-degrees of the factors. The polynomial degree will be called p-degree to avoid confusion – it is obviously not respected by the coproduct. We will use the notation $\mathcal{A}_i^{(n)}$ for the subspace of $\mathcal{A}$ of v-degree n and p-degree i, e.g., $\mathcal{A}_1^{(n)}$ is the linear span of the generators with n vertices.

3.2. Vector fields. The rôle of $\mathcal{U}$ is now played by $\mathcal{H}_R^*$, generated by $\{Z_T\}$, with T a rooted tree and we take the Z's dual to the ϕ's, in the sense of the previous section. Z_T is a left invariant vector field on G. The Lie algebra of such vector fields is found by computing, using (3), the pairing of $Z_A Z_B - Z_B Z_A$ with all basis functions $\{f^i\}$.

Example 1. Computation of $[Z_\bullet,\ Z_{\genfrac{}{}{0pt}{}{\bullet}{\bullet}}]$. We have

$$\tilde{\Delta}\Big(\substack{\bullet\\\bullet\\\bullet}\Big) = \bullet \otimes \genfrac{}{}{0pt}{}{\bullet}{\bullet} + \genfrac{}{}{0pt}{}{\bullet}{\bullet} \otimes \bullet, \qquad \tilde{\Delta}\Big(\genfrac{}{}{0pt}{}{\bullet}{\bullet\,\bullet}\Big) = 2\,\bullet \otimes \genfrac{}{}{0pt}{}{\bullet}{\bullet} + \bullet\bullet \otimes \bullet,$$
$$\tilde{\Delta}\Big(\bullet\genfrac{}{}{0pt}{}{\bullet}{\bullet}\Big) = \bullet \otimes \genfrac{}{}{0pt}{}{\bullet}{\bullet} + \genfrac{}{}{0pt}{}{\bullet}{\bullet} \otimes \bullet + \bullet\bullet \otimes \bullet + \bullet \otimes \bullet\bullet, \tag{12}$$

where $\tilde{\Delta}(\phi^T) \equiv \Delta(\phi^T) - \phi^T \otimes 1 - 1 \otimes \phi^T$. These are the only functions that contain the term $\bullet \otimes \genfrac{}{}{0pt}{}{\bullet}{\bullet}$ in their coproduct. We find therefore, using (3),

$$\Big\langle Z_\bullet Z_{\genfrac{}{}{0pt}{}{\bullet}{\bullet}}, \substack{\bullet\\\bullet\\\bullet} \Big\rangle = 1, \qquad \Big\langle Z_\bullet Z_{\genfrac{}{}{0pt}{}{\bullet}{\bullet}}, \genfrac{}{}{0pt}{}{\bullet}{\bullet\,\bullet} \Big\rangle = 2, \qquad \Big\langle Z_\bullet Z_{\genfrac{}{}{0pt}{}{\bullet}{\bullet}}, \bullet\genfrac{}{}{0pt}{}{\bullet}{\bullet} \Big\rangle = 1. \tag{13}$$

Similarly, one computes

$$\Big\langle Z_{\genfrac{}{}{0pt}{}{\bullet}{\bullet}} Z_\bullet, \substack{\bullet\\\bullet\\\bullet} \Big\rangle = 1, \qquad \Big\langle Z_{\genfrac{}{}{0pt}{}{\bullet}{\bullet}} Z_\bullet, \bullet\genfrac{}{}{0pt}{}{\bullet}{\bullet} \Big\rangle = 1, \tag{14}$$

the pairings with all other functions being zero. It follows that the only non-zero pairing of the commutator is

$$\Big\langle [Z_\bullet,\ Z_{\genfrac{}{}{0pt}{}{\bullet}{\bullet}}], \genfrac{}{}{0pt}{}{\bullet}{\bullet\,\bullet} \Big\rangle = 2. \tag{15}$$

But the element $2Z_{\genfrac{}{}{0pt}{}{\bullet}{\bullet\,\bullet}}$ of $\mathcal{U}$ has exactly the same pairings, therefore, in order for the inner product between $\mathcal{U}$ and $\mathcal{A}$ to be non-degenerate, one must set $[Z_\bullet,\ Z_{\genfrac{}{}{0pt}{}{\bullet}{\bullet}}] = 2Z_{\genfrac{}{}{0pt}{}{\bullet}{\bullet\,\bullet}}$.

Proceeding along these lines, one arrives at the general expression [7]

$$[Z_{T_1},\ Z_{T_2}] = \sum_T \left(n_{T_1 T_2}{}^T - n_{T_2 T_1}{}^T \right) Z_T \equiv \sum_T f_{T_1 T_2}{}^T Z_T, \tag{16}$$

where $n_{T_1 T_2}{}^T$ is the number of simple cuts on T that produce T_1, T_2, with T_2 containing the root of T (denoted by $n(T_1, T_2, T)$ in [6]) and the last equation defines the structure constants $f_{T_1 T_2}{}^T$ of the Lie algebra. We introduce, following [7], a $*$-operation among the Z's, defined by

$$Z_{T_1} * Z_{T_2} = n_{T_1 T_2}{}^T Z_T. \tag{17}$$

Notice that this is *not* the product in $\mathcal{U}$ but, nevertheless, it gives correctly the commutator when antisymmetrized (cf. (16)). The above Lie bracket conserves the number of vertices.

3.3. 1-forms. We turn now to LI 1-forms. Starting from (5) and using the particular form of the coproduct in (8), we find

$$\Pi_{\phi^T} = \sum_{C'} \phi^{S(P^{C'}(T))}\, d\phi^{R^{C'}(T)} = d\phi^T + \sum_{C} \phi^{S(P^{C}(T))}\, d\phi^{R^{C}(T)}. \tag{18}$$

For the MC equations we may use directly (7) and the comment that follows it to find

$$d\Pi_{\phi^T} = - \sum_{\text{simple } C} \Pi_{\phi^{P^C(T)}}\, \Pi_{\phi^{R^C(T)}}. \tag{19}$$

The restriction to simple cuts is possible since cuts that involve more than one edge produce non-linear terms in the first tensor factor of the coproduct and these are annihilated by Π. This is probably the easiest way to derive the structure constants.

Example 2. Maurer–Cartan equation for $\Pi_{\overset{\bullet}{\bullet\,\bullet}}$. Using (18) we find

$$\Pi_{\bullet} = d\bullet, \qquad \Pi_{\overset{\bullet}{\bullet}} = d\overset{\bullet}{\bullet} - \bullet d\bullet, \qquad \Pi_{\overset{\bullet}{\bullet\,\bullet}} = d\overset{\bullet}{\bullet\,\bullet} - 2\bullet d\overset{\bullet}{\bullet} + \bullet\bullet d\bullet. \tag{20}$$

Direct application of d to the above expression for $\Pi_{\overset{\bullet}{\bullet\,\bullet}}$, or use of (19), gives

$$d\Pi_{\overset{\bullet}{\bullet\,\bullet}} = -2\,\Pi_{\bullet}\Pi_{\overset{\bullet}{\bullet}}, \tag{21}$$

in agreement with the commutator $[Z_{\bullet},\ Z_{\overset{\bullet}{\bullet}}] = 2Z_{\overset{\bullet}{\bullet\,\bullet}}$ of Ex. 1.

General vector and 1-form fields are obtained as linear combinations of the above, with coefficients in $\mathcal{A}$.

4. Normal Coordinates

4.1. A new basis. We introduce new coordinates $\{\psi^A\}$ on G, defined by

$$\left\langle g, \psi^A \right\rangle = \alpha^A, \qquad \text{where} \quad g = e^{\alpha^A Z_A}, \tag{22}$$

i.e., the ψ's are normal coordinates centered at the origin and, like the ϕ's, are indexed by rooted trees. Of fundamental importance in the sequel will be the *canonical element* $\mathbf{C}$ (see, e.g., [4]), given by

$$\mathbf{C} = e_i \otimes f^i = e^{Z_A \otimes \psi^A}. \tag{23}$$

$\{e_i\}$ and $\{f^i\}$ above are dual bases of $\mathcal{U}$ and $\mathcal{A}$ respectively (see (4)). In contrast with (4), we fix now the $\{e_i\}$ to be $\{1,\ Z_A,\ Z_A Z_B,\ \dots\}$ and define the ψ's by the second equality above (the tensor product sign ensures that the Z's do not act on the ψ's). $\mathbf{C}$ may be regarded as an "indefinite group element" – when the ψ's get evaluated on some specific point g_0 of the group manifold, $\mathbf{C}$ becomes g_0. One may also view $\mathbf{C}$ as an "indefinite function" on the group – when the Z's get evaluated on some particular (analytic) ϕ_0, the resulting Taylor series delivers ϕ_0, i.e.,

$$\left\langle e^{Z_A \otimes \psi^A}, \mathrm{id} \otimes g_0 \right\rangle = g_0, \qquad \left\langle e^{Z_A \otimes \psi^A}, \phi_0 \otimes \mathrm{id} \right\rangle = \phi_0. \tag{24}$$

In the above, g_0, ϕ_0 stand for *any* element in the corresponding universal enveloping algebra, not just the generators. The second of (24) gives the relation between the two linear bases $\{f^i_{(\phi)}\}$ and $\{f^i_{(\psi)}\}$, generated by the ϕ's and the ψ's respectively. Indeed, taking $\phi_0 = \phi^A$ and expanding the exponential we find

$$\begin{aligned}\phi^A &= \sum_{m=0}^{\infty} \frac{1}{m!} \left\langle Z_{B_1} \dots Z_{B_m}, \phi^A \right\rangle \psi^{B_1} \dots \psi^{B_m} \\ &= \psi^A + \frac{1}{2} \left\langle Z_{B_1} Z_{B_2}, \phi^A \right\rangle \psi^{B_1} \psi^{B_2} + \dots .\end{aligned} \tag{25}$$

Lemma 1. *The change of linear basis in $\mathcal{A}$ generated by (25) is invertible.*

Proof. Notice that the linear part of $\phi^A(\psi)$ is ψ^A and also, that the above expansion preserves the v-degree. We choose a linear basis in $\mathcal{A}$ with the following ordering

$$\{ \underbrace{\phi^{\bullet}}_{v=1}, \underbrace{\phi^{\genfrac{}{}{0pt}{}{\bullet}{\bullet}}, \phi^{\bullet}\phi^{\bullet}}_{v=2}, \underbrace{\phi^{\genfrac{}{}{0pt}{}{\bullet}{\genfrac{}{}{0pt}{}{\bullet}{\bullet}}}, \phi^{\bullet\wedge\bullet}, \phi^{\genfrac{}{}{0pt}{}{\bullet}{\bullet}}\phi^{\bullet}, (\phi^{\bullet})^3}_{v=3}, \dots \}, \tag{26}$$

namely, in blocks of increasing v-degree and, within each block, non-decreasing p-degree. The above remarks then show that the matrix A, defined by

$$f^i_{(\phi)} = A^i{}_j f^j_{(\psi)}, \tag{27}$$

where $\{f^i_{(\psi)}\}$ is also ordered as in (26), is upper triangular, with units along the diagonal and hence invertible. □

Notice that A is in block-diagonal form, with each block A_v acting on $\mathcal{A}^{(v)}$, $v = 1, 2, \dots$. The computation of $\phi^A(\psi)$, via (25), reduces essentially to the evaluation of the inner product of ϕ^A with monomials in the Z's – this is facilitated by the following

Lemma 2. *The inner product $\left\langle Z_{B_1} \dots Z_{B_m}, \phi^A \right\rangle$ is given by*

$$\left\langle Z_{B_1} \dots Z_{B_m}, \phi^A \right\rangle = \left\langle Z_{B_1} * \dots * Z_{B_m}, \phi^A \right\rangle = n_{B_1 \dots B_m}{}^A, \tag{28}$$

where

$$n_{B_1 \dots B_m}{}^A = n_{B_1 R_1}{}^A \, n_{B_2 R_2}{}^{R_1} \dots n_{B_{m-1} B_m}{}^{R_{m-2}} \tag{29}$$

($Z_{B_1} * \dots * Z_{B_m}$ above is computed starting from the right, e.g., $Z_{B_1} * Z_{B_2} * Z_{B_3} \equiv Z_{B_1} * (Z_{B_2} * Z_{B_3})$).

Proof. We have

$$\left\langle Z_{B_1} \dots Z_{B_m}, \phi^A \right\rangle = \left\langle Z_{B_1} \otimes \dots \otimes Z_{B_m}, \Delta^{m-1}(\phi^A) \right\rangle. \tag{30}$$

In the above inner product, only the m-linear terms in $\Delta^{m-1}(\phi^A)$ contribute, since the Z's vanish on products and the unit function. One particular way of evaluating the $(m-1)$-fold coproduct is to apply Δ always on the rightmost tensor factor. It is then clear that, in this case, we may instead apply Δ_{lin}, since $\left(\prod_{j=1}^m \left(\text{id}^{\otimes j-1} \otimes \Delta\right)\right)(\phi^A)$ and $\left(\prod_{j=1}^m \left(\text{id}^{\otimes j-1} \otimes \Delta_{\text{lin}}\right)\right)(\phi^A)$ only differ by terms containing products of the ϕ's or units (this is only true if Δ_{lin} is applied in the rightmost factor). Notice now that the $*$-product of the Z's is dual to Δ_{lin},

$$\left\langle Z_{B_1} * Z_{B_2}, \phi^A \right\rangle = \left\langle Z_{B_1} \otimes Z_{B_2}, \Delta_{\text{lin}}(\phi^A) \right\rangle. \tag{31}$$

Repeated application of this equation and use of the definition of $*$, Eq. (17), completes the proof. □

A concise way to express the relation between the two sets of generators is via the $*$-*exponential* ($x \in \mathcal{U}_1$)

$$e_*^x \equiv \sum_{i=0}^{\infty} \frac{1}{i!} x^{*i} = \sum_{i=0}^{\infty} \frac{1}{i!} \underbrace{x * \cdots * x}_{i \text{ factors}}. \tag{32}$$

Combining (25) and (28) we find

$$e_*^{Z_A \otimes \psi^A} = Z_B \otimes \phi^B, \tag{33}$$

where the convention $(Z_A \otimes \psi^A) * (Z_B \otimes \psi^B) = Z_A * Z_B \otimes \psi^A \psi^B$ is understood and the sum on the r.h.s. starts with $1 \otimes 1$.

4.2. The Hopf structure. We derive now the Hopf data for the new basis. A standard property of $\mathbf{C}$ is

$$(\Delta \otimes \text{id})\mathbf{C} = \mathbf{C}_{13}\mathbf{C}_{23}, \qquad (\text{id} \otimes \Delta)\mathbf{C} = \mathbf{C}_{12}\mathbf{C}_{13}, \tag{34}$$

where, e.g., $\mathbf{C}_{13} \equiv e^{Z_A \otimes 1 \otimes \psi^A}$ – this is just the product-coproduct duality in (3). The second of (34) permits the calculation of the coproduct of the ψ's by applying the Baker–Cambell–Hausdorff (BCH) formula to the product on its r.h.s., $\Delta(\psi^A)$ is the coefficient of Z_A in the resulting single exponential

$$\begin{aligned}
&\exp\big(Z_A \otimes \Delta(\psi^A)\big) \\
&\quad = \exp\big(Z_A \otimes \psi^A \otimes 1\big)\exp\big(Z_B \otimes 1 \otimes \psi^B\big) \\
&\quad = \exp\Big(Z_A \otimes \psi^A \otimes 1 + Z_B \otimes 1 \otimes \psi^B + \frac{1}{2}[Z_A,\ Z_B] \otimes \psi^A \otimes \psi^B + \ldots\Big) \\
&\quad = \exp\Big\{Z_A \otimes \big(\psi^A \otimes 1 + 1 \otimes \psi^A + \frac{1}{2}f_{B_1B_2}{}^{A}\psi^{B_1} \otimes \psi^{B_2} + \ldots\big)\Big\},
\end{aligned} \tag{35}$$

so that

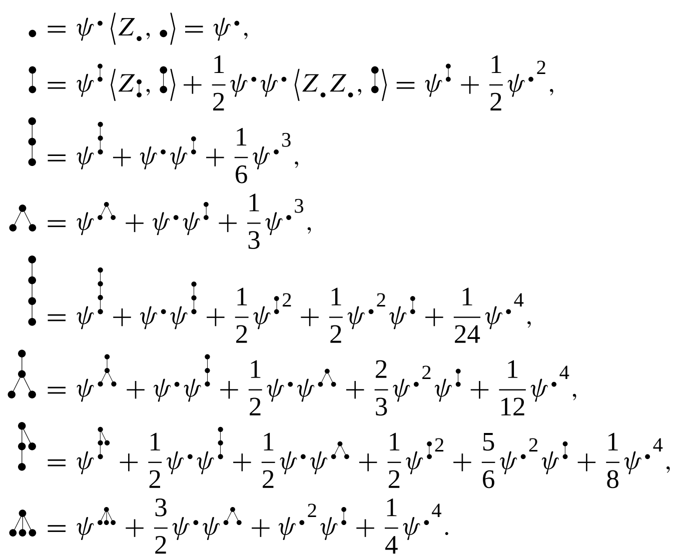
$$\Delta(\psi^A) = \psi^A \otimes 1 + 1 \otimes \psi^A + \frac{1}{2}f_{B_1B_2}{}^{A}\psi^{B_1} \otimes \psi^{B_2} + \ldots. \tag{36}$$

Higher terms in the coproduct can be computed by using a recursion relation for the BCH formula (see, e.g., Sect. 16 of [1]). The counit of all ψ^A vanishes. Although $\Delta(\psi^A)$ can be complicated, $S(\psi^A)$ never is. Using $\big\langle S(g),\, \psi^A\big\rangle = \big\langle g,\, S(\psi^A)\big\rangle$ and the fact that $S(g) = g^{-1}$, it is easily inferred that

$$S(\psi^A) = -\psi^A, \tag{37}$$

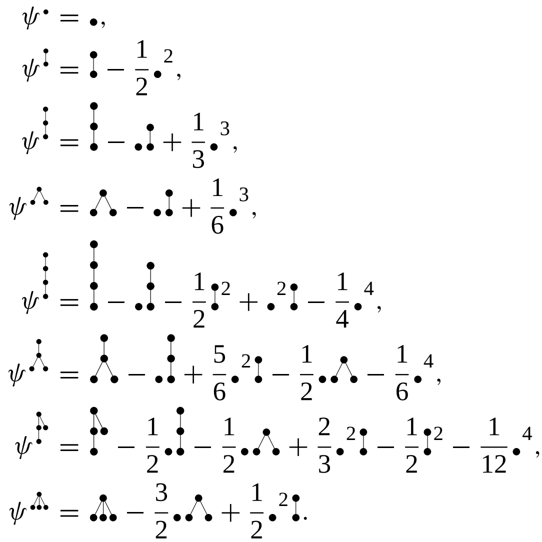
which extends as $S(p_r(\psi)) = (-1)^r p_r(\psi)$ on homogeneous polynomials of p-degree r. We see the first of the many advantages of working in the ψ-basis: the antipode is diagonal.

Example 3. Computation of $\psi^{(n)}$, $n \leq 4$. A straightforward application of (25) gives



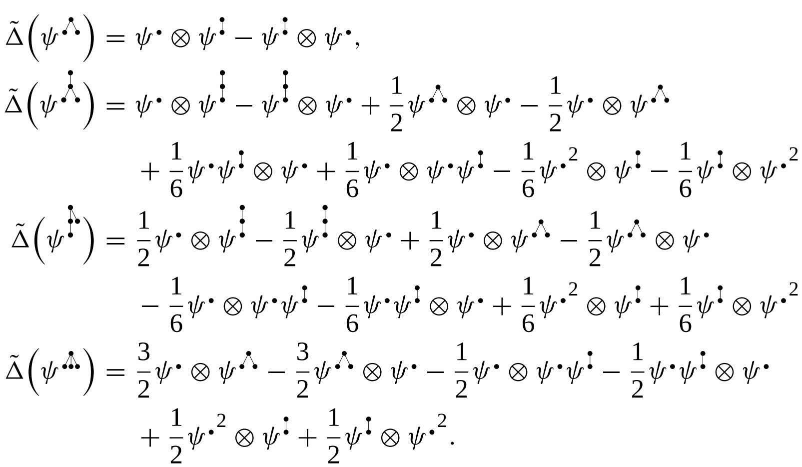
(38)

Inverting the above expressions we find



(39)

Concerning the coproduct, Eq. (36) shows that all ladder ψ's are primitive. For the rest of the ψ's, we get (omitting the primitive part)



(40)

One can easily verify that $S(\psi^A) = -\psi^A$.

5. Primitive Elements

We turn now to the study of the primitive elements of $\mathcal{A}$. These are of fundamental importance in any Hopf algebra, but acquire even more privileged status in our case, given their rôle in renormalization. Apart from this, they are also of interest in representation theory: given a primitive element $a \in \mathcal{A}$, $\Delta(a) = a \otimes 1_{\mathcal{A}} + 1_{\mathcal{A}} \otimes a$, one obtains a one-dimensional representation ρ_a of $\mathcal{U}$ via

$$\rho_a(x) \equiv \langle x, e^a \rangle . \tag{41}$$

Indeed, e^a is group-like, $\Delta(e^a) = e^a \otimes e^a$, so that

$$\rho_a(xy) \equiv \langle xy, e^a \rangle = \langle x \otimes y, e^a \otimes e^a \rangle = \rho_a(x)\rho_a(y). \tag{42}$$

Conversely, every one-dimensional representation of $\mathcal{U}$ is associated to some primitive element in $\mathcal{A}$.

Primitive elements are typically rare, but the algebra of rooted trees is quite exceptional in this respect: there is an infinite number of them in $\mathcal{A}$, with a non-trivial index set. We start our discussion with the easiest case, that of the ladder generators, for which our Theorem 1 below supplies a complete answer. We then turn to the considerably more complicated general case which Theorem 2 reduces to the problem of finding all closed LI 1-forms on G.

5.1. Ladder generators. We consider the subalgebra $\mathcal{T}$ of $\mathcal{H}_R$ generated by the ladder generators T_n, where n counts the number of vertices. Their coproduct is

$$\Delta(T_n) = \sum_{k=0}^{n} T_k \otimes T_{n-k}, \tag{43}$$

making $\mathcal{T}$ a sub-Hopf algebra of $\mathcal{H}_R$ (notice though that for ϕ not in $\mathcal{T}$, $\Delta(\phi)$ may involve terms in $\mathcal{T} \otimes \mathcal{T}$). Experimenting a little we find that, for the first few n's, each T_n gives rise to a primitive $P^{(n)}$. The general case is handled by the following

Theorem 1. *To each ladder generator $T_n, n = 1, 2, \ldots,$ corresponds a primitive element $P^{(n)}$, with T_n as its linear part, given by*

$$\boxed{P^{(n)} = \frac{1}{n!}\frac{\partial^n}{\partial x^n} \log\left(\sum_{m=0}^{\infty} T_m x^m\right)\bigg|_{x=0}.} \tag{44}$$

Proof. Consider the algebra $\mathcal{F}$ of formal power series $f(x) = \sum_{n=0}^{\infty} c_n x^n$, $c_0 = 1$, with the usual product. Define a basis $\{\xi_n,\ n = 0, 1, 2, \ldots\}$ of $\mathcal{F}^*$, the dual of $\mathcal{F}$, via

$$\langle \xi_n, f(x) \rangle = c_n, \tag{45}$$

i.e., ξ_n reads off the coefficient of x^n in f and $\xi_0 = 1$. For $f''(x) = f'(x) f(x)$ we have[2]

$$f''(x) = \sum_{n=0}^{\infty} c''_n x^n, \qquad c''_n = \sum_{k=0}^{n} c'_k c_{n-k}, \tag{46}$$

which implies the coproduct

$$\Delta(\xi_n) = \sum_{k=0}^{n} \xi_k \otimes \xi_{n-k} \tag{47}$$

[2] Notice that primes only distinguish functions here, they do not denote differentiation.

in $\mathcal{F}^*$. Endowing $\mathcal{F}^*$ with a commutative product, we arrive at the isomorphism $\mathcal{F}^* \cong \mathcal{T}$, as Hopf algebras, with $\xi_n \leftrightarrow T_n$. Define a new basis $\{\sigma_n,\ n = 0, 1, 2, \dots\}$ in $\mathcal{F}^*$ by

$$\langle \sigma_r, f(x) \rangle = \tilde{c}_r, \qquad \text{with} \quad f(x) = e^{\sum_{r=1}^{\infty} \tilde{c}_r x^r} \tag{48}$$

and $\sigma_0 = 1$. Then

$$f''(x) = e^{\sum_{r=1}^{\infty} \tilde{c}''_r x^r}, \qquad \text{with} \quad \tilde{c}''_r = \tilde{c}'_r + \tilde{c}_r, \tag{49}$$

implying the coproduct $\Delta(\sigma_r) = \sigma_r \otimes 1 + 1 \otimes \sigma_r$. The σ's, under the above isomorphism, correspond to the $P^{(n)}$ in $\mathcal{T}$. Solving the equation

$$e^{\sum_{r=1}^{\infty} P^{(r)} x^r} = \sum_{n=0}^{\infty} T_n x^n \tag{50}$$

for $P^{(r)}$, one arrives at (44). □

We read off $P^{(n)}$, for the first few values of n, as the coefficient of x^n in the Taylor series expansion

$$\begin{aligned}
\log\left(\sum_{n=0}^{\infty} T_n x^n\right) = {} & T_1\, x + \big(T_2 - \frac{1}{2}T_1^2\big)\, x^2 + \big(T_3 - T_1 T_2 + \frac{1}{3}T_1^3\big)\, x^3 \\
& + \big(T_4 - T_1 T_3 - \frac{1}{2}T_2^2 + T_1^2 T_2 - \frac{1}{4}T_1^4\big)\, x^4 \\
& + \big(T_5 - T_1 T_4 - T_2 T_3 + T_1^2 T_3 + T_1 T_2^2 - T_1^3 T_2 + \frac{1}{5}T_1^5\big)\, x^5 \\
& + \big(T_6 - T_1 T_5 - T_2 T_4 - \frac{1}{2}T_3^2 + T_1^2 T_4 + 2T_1 T_2 T_3 - T_1^3 T_3 \\
& + \frac{1}{3}T_2^3 - \frac{3}{2}T_1^2 T_2^2 + T_1^4 T_2 - \frac{1}{6}T_1^6\big)\, x^6 + \dots .
\end{aligned} \tag{51}$$

The polynomials $P^{(n)}(T_i)$ are known as *Schur polynomials*.

5.2. The general case. Given a closed LI 1-form α on G, there exists a linear combination $\phi^{i'}$ of the generators ϕ^A such that $\alpha = \Pi_{\phi^{i'}}$. Applying the inverse Poincaré lemma, we may write (locally)

$$\Pi_{\phi^{i'}} = d\psi^{i'}, \tag{52}$$

for some function $\psi^{i'}$ in $\mathcal{A}$. Requiring additionally that $\psi^{i'}$ vanish at the origin, $\epsilon(\psi^{i'}) = 0$, fixes the constant left arbitrary by (52) to zero. $\psi^{i'}$ can be expressed in terms of the ϕ's. Since $\Pi_{\phi^{i'}}$ reduces to $d\phi^{i'}$ at the origin, the linear part $\psi^{i'}_{\text{lin}}$ of $\psi^{i'}(\phi)$ is $\phi^{i'}$. But then $\Pi_{\phi^{i'}} = \Pi_{\psi^{i'}}$, since Π projects to the linear part. Comparing the r.h.s. of (52) with the general expression for a LI 1-form, Eq. (5), we conclude that $\psi^{i'}$ is primitive. Conversely, every primitive function $\psi^{i'}$ gives rise to a closed LI 1-form, $d\Pi_{\psi^{i'}} = dd\psi^{i'} = 0 = d\Pi_{\psi^{i'}_{\text{lin}}}$.

Equation (7), and the comment that follows it, show that $\Delta_{\text{lin}}(\phi^{i'})$ is symmetric under the interchange of its two tensor factors. This observation leads to a particularly simple way to identify primitive elements. One first looks for linear combinations $\phi^{i'}$ of the ϕ^A with symmetric $\Delta_{\text{lin}}(\phi^{i'})$ (notice that Δ_{lin} is given by simple cuts). The explicit expression for the corresponding primitive $\psi^{i'}$ then is given by the standard formula for the (local) potential of a closed form. We find that the result is simplified considerably due to the particular form of the coproduct of the ϕ^A, namely the linearity of $\Delta(\phi^A)$ in its second tensor factor.

Theorem 2. *Given $\phi^{i'} \in \mathcal{A}_1$, such that $d\Pi_{\phi^{i'}} = 0$. Then the element $\psi^{i'}$ of $\mathcal{A}$, given by*

$$\boxed{\psi^{i'} = -\Phi^{-1} \circ S(\phi^{i'}),} \tag{53}$$

is primitive and has $\phi^{i'}$ as its linear part (Φ above is the p-degree operator for the ϕ's, $\Phi(\phi^{A_1} \dots \phi^{A_r}) = r\phi^{A_1} \dots \phi^{A_r}$).

Proof. We apply the inverse Poincaré lemma to $\Pi_{\phi^{i'}}$. For a given v-degree n, only ϕ^A of v-degree up to n enter in the formulas – we denote them collectively by $\boldsymbol{x}$ (e.g., $S(\phi)(\boldsymbol{x})$ denotes the standard expression of $S(\phi)$ in terms of the ϕ^A while $S(\phi)(z\boldsymbol{x})$ denotes the same expression with every ϕ^A multiplied by z). Consider the family of diffeomorphisms $\varphi_t : \boldsymbol{x} \mapsto (1-t)\boldsymbol{x}$, $0 \le t \le 1$. Then φ_0^* is the identity map while φ_1^* is the zero map. The corresponding velocity field is

$$\boldsymbol{v} = \frac{d}{dt}\varphi_t(\boldsymbol{x}) = -\boldsymbol{x} \Rightarrow \boldsymbol{v}(\boldsymbol{y}, t) = -\frac{1}{1-t}\boldsymbol{y}, \tag{54}$$

where $\boldsymbol{y} = \varphi_t(\boldsymbol{x})$. We have[3]

$$\Pi_{\phi^{i'}}(\boldsymbol{x}) = \varphi_0^*\big(\Pi_{\phi^{i'}}(\varphi_0(\boldsymbol{x}))\big) - \varphi_1^*\big(\Pi_{\phi^{i'}}(\varphi_1(\boldsymbol{x}))\big) = \int_1^0 dt\, \frac{d}{dt}\varphi_t^*\big(\Pi_{\phi^{i'}}(\boldsymbol{y})\big). \tag{55}$$

However, $\frac{d}{dt}\varphi_t^* = \varphi_t^* L_{\boldsymbol{v}} = \varphi_t^*(d\,\mathbf{i}_{\boldsymbol{v}} + \mathbf{i}_{\boldsymbol{v}}\, d)$ and, taking into account the closure of $\Pi_{\phi^{i'}}$, we find

$$\Pi_{\phi^{i'}}(\boldsymbol{x}) = d \int_1^0 dt\, \varphi_t^*\big(\mathbf{i}_{\boldsymbol{v}}\, \Pi_{\phi^{i'}}(\boldsymbol{y})\big). \tag{56}$$

This is the inverse Poincaré lemma. We concentrate now on the action of $\mathbf{i}_{\boldsymbol{v}}$ on $\Pi_{\phi^{i'}}(\boldsymbol{y})$. We have

$$\mathbf{i}_{\boldsymbol{v}} = -\frac{1}{1-t}\, y^j\, \mathbf{i}_{\partial_{y^j}}, \qquad \Pi_{\phi^{i'}}(\boldsymbol{y}) = S(\phi^{i'}_{(1)})\, d\phi^{i'}_{(2)}(\boldsymbol{y}). \tag{57}$$

In this latter (implied) sum, all terms in the coproduct of $\phi^{i'}$ appear except the first one, $\phi^{i'} \otimes 1$, which is annihilated by d. Notice now that $y^j\, \mathbf{i}_{\partial_{y^j}} dy^i = y^i$. Since $\Delta(\phi^{i'})$ is linear in its second factor we conclude that

$$y^j \mathbf{i}_{\partial_{y^j}} S(\phi^{i'}_{(1)})\, d\phi^{i'}_{(2)}(\boldsymbol{y}) = S(\phi^{i'}_{(1)})\, \phi^{i'}_{(2)}(\boldsymbol{y}) - S(\phi^{i'})(\boldsymbol{y}) = -S(\phi^{i'})(\boldsymbol{y}). \tag{58}$$

[3] We ignore in the sequel the singularity of $\boldsymbol{v}$ at $t = 1$ – it is easily shown to be harmless.

Substituting back into (56) and putting $1 - t \equiv z$ we find

$$\Pi_{\phi^{i'}}(\boldsymbol{x}) = -d \int_0^1 \frac{dz}{z} S(\phi^{i'})(z\boldsymbol{x}), \tag{59}$$

which, upon performing the integration over z, gives $\Pi_{\phi^{i'}} = -d\Phi^{-1} \circ S(\phi^{i'})$. The remarks preceding the theorem complete the proof. □

5.3. The lower central series and k-primitiveness. We extend here the notion of primitiveness to that of k-primitiveness. Our starting point is our BCH-based prescription for calculating the coproduct of the ψ's, Eq. (36). Suppose we identify all generators $Z_i^{[1]}$ of $\mathcal{G}$ that cannot be written as commutators (the $Z_i^{[1]}$ are, in general, linear combinations of the Z_A). Then we may perform a linear change of basis in $\mathcal{G}$ and split the generators into two classes, one made up of the above $Z_i^{[1]}$ and the other spanning the complement – we denote the latter by $\{Z_{i'}\}$. Writing the canonical element in the new basis,

$$\mathbf{C} = e^{Z_i^{[1]} \otimes \psi_{[1]}^i + Z_{i'} \otimes \psi^{i'}}, \tag{60}$$

we are led to the identification of the $\psi_{[1]}^i$ with the primitive ψ's. This is so since, in the BCH formula, the $Z_i^{[1]}$ are never produced by the commutators, so that the only contribution to $\Delta(\psi_{[1]}^i)$ is the primitive part. Consider now the lower central series of $\mathcal{G}$, consisting of the series of subspaces $\mathcal{G}^{[1]}, \mathcal{G}^{[2]}, \ldots$. A particular Z in $\mathcal{G}$ belongs to $\mathcal{G}^{[k]}$ if it can be written as a $(k-1)$-nested commutator. This implies that if Z belongs to $\mathcal{G}^{[k]}$, it also belongs to all $\mathcal{G}^{[r]}$, with $r < k$. This is the standard definition of $\mathcal{G}^{[k]}$ – we actually need a slightly modified one, according to which *Z belongs only to the $\mathcal{G}^{[k]}$ with the maximum k*. With this definition, $\mathcal{G}^{[k]} \cap \mathcal{G}^{[r]} = \emptyset$ whenever $k \neq r$. We may now perform a linear change of basis in $\mathcal{G}$ such that each generator $Z_i^{[k]}$ in the new basis belongs to $\mathcal{G}^{[k]}$. Writing the canonical element in the form

$$\mathbf{C} = e^{Z_i^{[k]} \otimes \psi_{[k]}^i}, \tag{61}$$

defines the *k-primitiveness* for the $\psi_{[k]}^i$ dual to the above $Z_i^{[k]}$. Since the $Z_i^{[k]}$ are linear combinations of the Z_A, the $\psi_{[k]}^i$ will be linear combinations of the ψ^A. $\mathcal{A}$ splits accordingly to a direct sum, $\mathcal{A} = \bigoplus_{k=1}^{\infty} \mathcal{A}^{[k]}$ – the primitive ψ's, in particular, span $\mathcal{A}^{[1]}$. Notice that ψ's with n vertices may belong to $\mathcal{G}^{[k]}$ with $k \leq n-1$. This is so because the "longest" nested commutator with n vertices is $[Z_\bullet, [Z_\bullet, \ldots [Z_\bullet, Z_{\bullet}^{\bullet}]] \ldots]$, with $n-2$ entries of $Z_\bullet$.

The above concept of k-primitiveness arose naturally in our study of the primitive $\psi_{[1]}^i$. Some time afterwards, we became aware of Ref. [3], where a concept of k-primitiveness is also defined, as follows: given an element χ of $\mathcal{A}$, one computes successive powers of the coproduct, $\Delta^k(\chi)$. There is a minimum k for which all terms in $\Delta^k(\chi)$ contain a unity in at least one of the tensor factors – this defines the k-primitiveness of χ. Our definition is intrinsically defined only on the generators $\psi_{[k]}^i$, while the above makes sense in all of $\mathcal{A}$. We now show that, for $\psi_{[k]}^i$, the two definitions coincide.

Lemma 3. *The minimum value of r for which $\Delta^r(\psi_{[k]}^i)$ contains at least one unit tensor factor in each of its terms, is $r = k$.*

Proof. The various powers of the coproduct of $\psi^i_{[k]}$ can be computed by iteration of the second of (34),

$$\Delta^{r-1}(\psi^i_{[k]}) = \text{coeff. of } \; Z^{[k]}_i \; \text{ in } \; \log\Big(\mathbf{C}_{01}\mathbf{C}_{02}\ldots\mathbf{C}_{0r}\Big). \tag{62}$$

This shows that in $\Delta^k(\psi^i_{[k]})$, the $(k+1)$-linear term can only be produced by the k-nested commutator

$$[Z_{i_1},\,[Z_{i_2},\,\ldots[Z_{i_k},\,Z_{i_{k+1}}]]\ldots]\otimes\psi^{i_1}\otimes\ldots\otimes\psi^{i_{k+1}}.$$

The latter, however, has no $Z^{[k]}_i$ component, since $Z^{[k]}_i$ can be written as a $(k-1)$-nested commutator at most. It is also clear, for the same reason, that there are no terms of higher p-degree in the ψ's, as those would correspond to even longer nested commutators. $\Delta^k(\psi^i_{[k]})$ then must have at least one unit tensor factor in each of its terms. On the other hand, the k-linear term in $\Delta^{k-1}(\psi^i_{[k]})$ is not zero, because, by definition, the corresponding $(k-1)$-nested commutator has a $Z^{[k]}_i$ component. □

As shown in [3], the k-degree satisfies

$$\deg_k(\psi^i_{[k_1]}\psi^j_{[k_2]}) = k_1 + k_2. \tag{63}$$

We use the two definitions of the k-degree interchangeably in what follows. We may now clarify the relation between the primitive elements given by the inverse Poincaré formula, Eq. (53), and the ones introduced above via the lower central series of $\mathcal{G}$.

Lemma 4. *Given $\phi^{i'} = c^{i'}{}_A\phi^A$, with $c^{i'}{}_A$ constants, such that $d\Pi_{\phi^{i'}} = 0$. Then the primitive element $\psi^{i'}$ of (53) is equal to $c^{i'}{}_A\psi^A$, i.e.,*

$$\psi^{i'} = -\Phi^{-1}\circ S(\phi^{i'}) = c^{i'}{}_A\psi^A. \tag{64}$$

All primitive elements of $\mathcal{A}$ can be obtained in this form.

Proof. Any linear combination of the $\psi^i_{[1]}$ is primitive, while (sums of) products of them are not, due to (63). Therefore, the $\psi^i_{[1]}$ constitute a linear basis in the vector space of primitive elements of $\mathcal{A}$. To the given $\phi^{i'}$, Eq. (53) associates a primitive element $\psi^{i'}$, with $\phi^{i'}$ as its linear part. The unique linear combination of the ψ^A (and, hence, of the $\psi^i_{[1]}$) with this linear part is $\psi^{i'} = c^{i'}{}_A\psi^A$. □

We give an example illustrating the above.

Example 4. Construction of $\mathcal{G}^{(n)[k]}$, $\mathcal{A}^{(n)[k]}$, for $n \le 4$. To identify the generators of $\mathcal{G}^{(n)[k]}$, we construct all $(k-1)$-nested commutators with n vertices – $\mathcal{G}^{(n)[1]}$ is determined as the complement of $\bigoplus_{k=2}^{n-1}\mathcal{G}^{(n)[k]}$ in $\mathcal{G}^{(n)}$ (below we use the orthogonal complement but this is not essential, one simply has to complete the basis of the Z's). This gives a matrix that effects the transition from the basis $\{Z_A\}$, indexed by rooted trees, to the basis $\{Z^{[k]}_i\}$, of definite k-primitiveness. The inverse matrix then gives the $\psi^i_{[k]}$ in terms of the ψ^A.

$\mathcal{G}^{(1)[1]} = \mathcal{G}^{(1)}$ is generated by $Z_{\bullet}$. $\mathcal{G}^{(2)[1]} = \mathcal{G}^{(2)}$ is generated by $Z_{\substack{\bullet\\\bullet}}$, since the only commutator with two vertices, $[Z_{\bullet},\ Z_{\bullet}]$ is zero. For $n = 3$, we have the only non-zero commutator[4] $Z_1^{(3)[2]} \equiv [Z_{\bullet},\ Z_{\substack{\bullet\\\bullet}}] = 2Z_{\substack{\bullet\\\bullet\,\bullet}}$. The complement in $\mathcal{G}^{(3)}$ is spanned by $Z_1^{(3)[1]} = Z_{\substack{\bullet\\\bullet\\\bullet}}$. Next we look at the case $n = 4$. We find the only non-zero commutators

$$[Z_{\bullet},\ Z_{\substack{\bullet\\\bullet\\\bullet}}] = (0, 2, 1, 0) \equiv Z_1^{(4)[2]}, \qquad [Z_{\bullet},\ Z_{\substack{\bullet\\\bullet\,\bullet}}] = (0, -1, 1, 3) \equiv Z_1^{(4)[3]}, \tag{65}$$

in the basis $\left\{Z_{\substack{\bullet\\\bullet\\\bullet\\\bullet}},\ Z_{\substack{\bullet\\\bullet\\\bullet\,\bullet}},\ Z_{\substack{\bullet\\\bullet\,\bullet\\\bullet\ }},\ Z_{\substack{\bullet\\\bullet\bullet\bullet}}\right\}$. The orthogonal complement in $\mathcal{G}^{(4)}$ is spanned by

$$Z_1^{(4)[1]} = (1, 0, 0, 0), \qquad Z_2^{(4)[1]} = (0, 1, -2, 1). \tag{66}$$

Writing the above change of basis symbolically as $Z_i^{[k]} = MZ_A$, with M a matrix of numerical coefficients, the dual change of basis for the ψ's is given by $\psi^i_{[k]} = \psi^A M^{-1}$. We find

$$\psi^1_{(1)[1]} = \psi^{\bullet}, \qquad \psi^1_{(2)[1]} = \psi^{\substack{\bullet\\\bullet}}, \qquad \psi^1_{(3)[1]} = \psi^{\substack{\bullet\\\bullet\\\bullet}}, \qquad \psi^1_{(3)[2]} = \frac{1}{2}\psi^{\substack{\bullet\\\bullet\,\bullet}}, \tag{67}$$

while, for $n = 4$,

$$\begin{aligned}
\psi^1_{(4)[1]} &= \psi^{\substack{\bullet\\\bullet\\\bullet\\\bullet}},\\
\psi^2_{(4)[1]} &= \frac{1}{6}\psi^{\substack{\bullet\\\bullet\\\bullet\,\bullet}} - \frac{1}{3}\psi^{\substack{\bullet\\\bullet\,\bullet\\\bullet\ }} + \frac{1}{6}\psi^{\substack{\bullet\\\bullet\bullet\bullet}},\\
\psi^1_{(4)[2]} &= \frac{7}{18}\psi^{\substack{\bullet\\\bullet\\\bullet\,\bullet}} + \frac{2}{9}\psi^{\substack{\bullet\\\bullet\,\bullet\\\bullet\ }} + \frac{1}{18}\psi^{\substack{\bullet\\\bullet\bullet\bullet}},\\
\psi^1_{(4)[3]} &= -\frac{1}{18}\psi^{\substack{\bullet\\\bullet\\\bullet\,\bullet}} + \frac{1}{9}\psi^{\substack{\bullet\\\bullet\,\bullet\\\bullet\ }} + \frac{5}{18}\psi^{\substack{\bullet\\\bullet\bullet\bullet}}.
\end{aligned} \tag{68}$$

Referring to, e.g., $\psi^2_{(4)[1]}$, one easily verifies that

$$\phi^{i'} = \frac{1}{6}\phi^{\substack{\bullet\\\bullet\\\bullet\,\bullet}} - \frac{1}{3}\phi^{\substack{\bullet\\\bullet\,\bullet\\\bullet\ }} + \frac{1}{6}\phi^{\substack{\bullet\\\bullet\bullet\bullet}} \tag{69}$$

has symmetric Δ_{lin} and, when inserted in (53), delivers $\psi^2_{(4)[1]}$.

To continue the above construction to the cases $n = 5, 6$, we developed a REDUCE program, incorporating some of the procedures of [2]. The numbers $P_{n,k}$ of k-primitive ψ's with $n \leq 6$ vertices that we find coincide with the ones in Table 4 of [3], as expected. In what refers to the primitive ψ's, the procedure presented above, starting with ϕ's with symmetric Δ_{lin} and then using (53), should be considerably more efficient than the one used in [3] – it would be interesting to quantify this statement. Notice that an equivalent procedure involves expanding the primitive ψ's as $\psi_{[1]} = c_A \psi^A$ and then determining the constants c_A from the set of equations $f_{RS}{}^{T}\left\langle Z_T,\, \psi_{[1]}\right\rangle = 0$ (the latter is the statement that $\psi_{[1]}$ is invariant under the coadjoint coaction).

[4] We remind the reader of our notation: $Z_i^{(n)[k]}$ is the i^{th} element in the subspace of k-primitive, n-vertex Z's. The same notation is used for the ψ's, with the position of the indices (upper–lower) interchanged.

6. Normal Coordinates and Toy Model Renormalization

We turn now to what, in some sense, is our main objective, namely, the application of the formalism presented so far in the problem of renormalization in perturbative quantum field theory. The scope of our considerations in this section can only be modest, since realistic quantum field theories involve rooted trees with an infinite number of decorations. Nevertheless, a toy model exists (see [10]) that realizes the ϕ^A as nested divergent integrals, regulated by a parameter ϵ. We find this an extremely useful construct that captures many of the most important features of realistic renormalization – again, we refer the reader to [10, 6] for a detailed presentation. What we are interested in here, is the rôle of the new coordinates ψ in the renormalization of divergent quantities. We start with a brief review of the basics.

6.1. The toy model. The elementary divergence in the toy model we deal with is given by the integral

$$I_1(c;\ \epsilon) = \int_0^\infty dy\,\frac{y^{-\epsilon}}{y+c}, \tag{70}$$

which diverges as ϵ goes to zero. c above will be referred to as the *external parameter* of the integral. We associate the function $\phi^{\bullet}$ with $I_1(c;\ \epsilon)$. To the function $\phi^{\substack{\bullet\\\bullet}}$ corresponds the nested integral

$$I_2(c;\ \epsilon) = \int_0^\infty dy_1\,\frac{y_1^{-\epsilon}}{y_1+c} I_1(y_1;\ \epsilon) = \int_0^\infty dy_1\,\frac{y_1^{-\epsilon}}{y_1+c}\int_0^\infty dy_2\,\frac{y_2^{-\epsilon}}{y_2+y_1}. \tag{71}$$

Notice that the external parameter of the subdivergence I_1 is y_1. To $\phi^{\substack{\bullet\\\bullet\\\bullet}}$, $\phi^{\substack{\bullet\\\bullet\,\bullet}}$ correspond, respectively,

$$I_{3,1}(c;\epsilon) = \int_0^\infty dy_1\,\frac{y_1^{-\epsilon}}{y_1+c} I_2(y_1;\epsilon), \qquad I_{3,2}(c;\epsilon) = \int_0^\infty dy_1\,\frac{y_1^{-\epsilon}}{y_1+c}\big(I_1(y_1;\epsilon)\big)^2, \tag{72}$$

it should be clear how this assignment extends to all ϕ^A. In this way, each ϕ^A can be associated with the Laurent series in ϵ that corresponds to its associated integral, e.g.

$$\phi^{\bullet} = \int_0^\infty dy\,\frac{y^{-\epsilon}}{y+c} = \frac{\pi}{\sin(\pi\epsilon)}c^{-\epsilon} = \frac{1}{\epsilon} - a + \mathcal{O}(\epsilon), \tag{73}$$

where $a \equiv \log(c)$ and, similarly (using MAPLE),

$$\begin{aligned}
\phi^{\substack{\bullet\\\bullet}} &= \frac{1}{2\epsilon^2} - \frac{a}{\epsilon} + a^2 + \frac{5\pi^2}{12} + \mathcal{O}(\epsilon),\\
\phi^{\substack{\bullet\\\bullet\\\bullet}} &= \frac{1}{6\epsilon^3} - \frac{a}{2\epsilon^2} + \Big(\frac{3a^2}{4} + \frac{7\pi^2}{18}\Big)\frac{1}{\epsilon} - \frac{a}{12}\big(9a^2 + 14\pi^2\big) + \mathcal{O}(\epsilon),\\
\phi^{\substack{\bullet\\\bullet\,\bullet}} &= \frac{1}{3\epsilon^3} - \frac{a}{\epsilon^2} + \Big(\frac{3a^2}{2} + \frac{11\pi^2}{18}\Big)\frac{1}{\epsilon} - \frac{a}{6}\big(9a^2 + 11\pi^2\big) + \mathcal{O}(\epsilon),\\
\phi^{\substack{\bullet\\\bullet\\\bullet\\\bullet}} &= \frac{1}{24\epsilon^4} - \frac{a}{6\epsilon^3} + \Big(\frac{a^2}{3} + \frac{5\pi^2}{24}\Big)\frac{1}{\epsilon^2} - \frac{a}{18}\Big(8a^2 + 15\pi^2\Big)\frac{1}{\epsilon} + \mathcal{O}(\epsilon^0),
\end{aligned} \tag{74}$$

$$\phi = \frac{1}{12\epsilon^4} - \frac{a}{3\epsilon^3} + \Big(\frac{2a^2}{3} + \frac{3\pi^2}{8}\Big)\frac{1}{\epsilon^2} - \frac{a}{18}\Big(16a^2 + 27\pi^2\Big)\frac{1}{\epsilon} + \mathcal{O}(\epsilon^0),$$
$$\phi = \frac{1}{8\epsilon^4} - \frac{a}{2\epsilon^3} + \Big(a^2 + \frac{11\pi^2}{24}\Big)\frac{1}{\epsilon^2} - \frac{a}{6}\Big(8a^2 + 11\pi^2\Big)\frac{1}{\epsilon} + \mathcal{O}(\epsilon^0),$$
$$\phi = \frac{1}{4\epsilon^4} - \frac{a}{\epsilon^3} + \Big(2a^2 + \frac{19\pi^2}{24}\Big)\frac{1}{\epsilon^2} - \frac{a}{6}\Big(16a^2 + 19\pi^2\Big)\frac{1}{\epsilon} + \mathcal{O}(\epsilon^0), ,$$

and so on. It is easily seen that ϕ's with n vertices give rise to Laurent series with leading pole of order n. The process of renormalization assigns to each ϕ^A a finite "renormalized" value $\phi^A_{\mathcal{R}}$ (see, e.g., [5]). In Hopf algebraic terms, the latter is given by [2]

$$\phi^A_{\mathcal{R}} = S_{\mathcal{R}}\big(\phi^A_{(1)}\big)\phi^A_{(2)}, \tag{75}$$

where the *twisted antipode* $S_{\mathcal{R}}$ is defined recursively by

$$S_{\mathcal{R}}\big(\phi^A\big) = -\mathcal{R}\big(\phi^A\big) - \mathcal{R}\Big(S_{\mathcal{R}}\big(\phi^A_{(1')}\big)\phi^A_{(2')}\Big). \tag{76}$$

$\mathcal{R}$ above is a *renormalization map* that we choose here to give the pole part of its argument, evaluated at the external parameter equal to 1, e.g., $\mathcal{R}\big(\phi\big) = 1/2\epsilon^2$ (compare with the first of (74)). The primed sum in the second term of (76) excludes the primitive part of the coproduct. The magic of renormalization lies in the fact that, for any ϕ^A, the renormalized $\phi^A_{\mathcal{R}}$ in (75) has no poles in ϵ – what makes this statement non-trivial is that all terms subtracted iteratively from ϕ^A, to give $\phi^A_{\mathcal{R}}$, are independent of external parameters. We conclude our brief review with the following statement, proven in [11]: if $\mathcal{R}$ satisfies the *multiplicative constraint*

$$\mathcal{R}(xy) - \mathcal{R}\big(\mathcal{R}(x)y\big) - \mathcal{R}\big(x\mathcal{R}(y)\big) + \mathcal{R}(x)\mathcal{R}(y) = 0, \tag{77}$$

then $S_{\mathcal{R}}$ is multiplicative, $S_{\mathcal{R}}(xy) = S_{\mathcal{R}}(x)S_{\mathcal{R}}(y)$ – our choice of $\mathcal{R}$ above does satisfy (77).

6.2. Renormalization in the ψ-basis. For a given number n of vertices, the renormalization of every generator ϕ^A gives rise to 2^n counterterms, for a total of $r_n 2^n$, where r_n is the number of rooted trees with n vertices. To renormalize the ψ's, one can always express them in terms of the ϕ's and then proceed as above. However, for renormalization schemes $\mathcal{R}$ that satisfy (77), a much more efficient possibility arises. Equation (75), in this case, is valid for *any* function in $\mathcal{A}$, and, in particular, for the ψ's. Notice that although the action of the antipode S is trivial on the ψ^A, that of the twisted antipode $S_{\mathcal{R}}$ is not, in general. The advantage of working in the basis $\{\psi^i_{[k]}\}$ is that the complexity of the renormalization of a generator $\psi^i_{(n)[k]}$ is governed by k, not n, which entails, in general, significant savings. As an extreme example, a primitive ψ with one hundred vertices is renormalized by a simple subtraction – this should be compared with the 2^{100} counterterms necessary for the renormalization of each of the $\phi_{(100)}$'s. How significant can the savings be in, e.g., CPU time, depends on the distribution of the $\psi^i_{(n)}$ in the various k-classes. As proved in [3], the numbers $P_{n,k}$ of k-primitive ψ's with n vertices are generated by

$$P_k(x) \equiv \sum_{n=1}^{\infty} P_{n,k}x^n = \sum_{s|k} \frac{\mu(s)}{k}\Big(1 - \prod_{n=1}^{\infty}\big(1 - x^{ns}\big)^{r_n}\Big)^{k/s}, \tag{78}$$

a rather non-trivial result. The sum in the r.h.s. above extends over all divisors s of k, including 1 and k. $\mu(s)$ is the Möbius function, equal to zero, if s is divisible by a square, and to $(-1)^p$, if s is the product of p distinct primes ($\mu(1) \equiv 1$). Of particular interest to us is the asymptotic behavior of $P_{n,k}$, for large values of n [3],

$$f_k \equiv \lim_{n\to\infty} \frac{P_{n,k}}{r_n} = \frac{1}{c}\left(1 - \frac{1}{c}\right)^{k-1}, \tag{79}$$

where $c = 2.95\ldots$ is the Otter constant. This is encouraging, as the population of the CPU-intensive high-k ψ's is seen to be exponentially suppressed. A realistic estimate of the complexity of renormalization in the ψ-basis is outside the scope of this article, as it would probably entail implementation-dependent parameters. Nevertheless, we attempt a first-order estimation by assigning a computational cost of 2^k to a k-primitive ψ, while the $\phi_{(n)}$ are assigned the cost 2^n. The ratio of the total costs of renormalizing all generators with n vertices in the two bases then is

$$\rho_n = \frac{r_n 2^n}{\sum_{k=1}^{n-1} P_{n,k} 2^k} \approx (c-2)\left(\frac{c}{c-1}\right)^{n-1}, \tag{80}$$

with $\rho_{33} \approx 6\times10^5$ making the difference between a week and a second. We consider (80) as a loose upper bound on the potential savings.

Another feature of the ψ's that is worth pointing out is their toy model pole structure. As mentioned above, each of the $\phi^A_{(n)}$ corresponds to a Laurent series with maximal pole order n. We find that the behavior of the $\psi^i_{(n)}$ is much milder. We list the series expansion of the first few ψ^A, which should be compared with the analogous expressions for the ϕ^A, Eq. (74),

$$\begin{aligned}
\psi^{\bullet} &= \frac{1}{\epsilon} - a + \mathcal{O}(\epsilon),\\
\psi^{\substack{\bullet\\\bullet}} &= \frac{\pi^2}{4} + \mathcal{O}(\epsilon),\\
\psi^{\substack{\bullet\\\bullet\\\bullet}} &= \frac{\pi^2}{18\epsilon} - \frac{\pi^2 a}{6} + \mathcal{O}(\epsilon),\\
\psi^{\substack{\bullet\\\bullet\,\bullet}} &= \frac{7\pi^2}{36\epsilon} - \frac{7\pi^2 a}{12} + \mathcal{O}(\epsilon),\\
\psi^{\substack{\bullet\\\bullet\\\bullet\\\bullet}} &= \frac{\pi^4}{8} + \mathcal{O}(\epsilon),\\
\psi^{\substack{\bullet\\\bullet\\\bullet\,\bullet}} &= \frac{19\pi^4}{72} + \mathcal{O}(\epsilon),\\
\psi^{\substack{\bullet\\\bullet\,\bullet\\\bullet}} &= \frac{\pi^2}{24\epsilon^2} - \frac{\pi^2 a}{6\epsilon} + \mathcal{O}(\epsilon^0),\\
\psi^{\substack{\bullet\\\bullet\bullet\bullet}} &= \frac{\pi^2}{12\epsilon^2} - \frac{\pi^2 a}{3\epsilon} + \mathcal{O}(\epsilon^0).
\end{aligned} \tag{81}$$

Notice that, e.g., the primitive $\psi^{\substack{\bullet\\\bullet\\\bullet\\\bullet}}$ is actually finite, as is $\psi^{\substack{\bullet\\\bullet\\\bullet\,\bullet}}$ which is not primitive. We emphasize that $\left(\psi^{\substack{\bullet\\\bullet\\\bullet\,\bullet}}\right)_{\mathcal{R}}$ is still given by (75) (with $\phi^a \to \psi^{\substack{\bullet\\\bullet\\\bullet\,\bullet}}$) and does not coincide with the finite ψ (see Ex. 5 below). The other two $\psi^i_{(4)}$ are of order $1/\epsilon^2$, even though they have $\mathcal{G}^{[3]}$ components. These initial observations point to a general feature of the ψ's: the pole order does not specify the complexity of their renormalization, as is the case with the ϕ's. The cancellations of the higher-order poles observed point to rather non-trivial underlying combinatorics that, we believe, deserve further investigation.

The series expansion of the $\psi^i_{(n)[k]}$ is

$$
\begin{aligned}
\psi^2_{(4)[1]} &= \frac{\pi^4}{48} + \mathcal{O}(\epsilon), \\
\psi^1_{(4)[2]} &= \frac{\pi^2}{72\epsilon^2} - \frac{\pi^2 a}{18\epsilon} + \mathcal{O}(\epsilon^0), \\
\psi^1_{(4)[3]} &= \frac{\pi^2}{36\epsilon^2} - \frac{\pi^2 a}{9\epsilon} + \mathcal{O}(\epsilon^0)
\end{aligned}
\tag{82}
$$

(the rest are essentially identical to the ψ^A). We also point out that some of the $n = 6$ primitive ψ's are of order $1/\epsilon^3$ – nevertheless, the coefficients of all poles are independent of c and their renormalization is accomplished by a simple subtraction, in agreement with (75).

Example 5. Renormalization of $\psi^2_{(4)[1]}$, $\psi^1_{(4)[2]}$, ψ. For the primitive $\psi^2_{(4)[1]}$, Eqs. (76), (82) give

$$
S_{\mathcal{R}}\big(\psi^2_{(4)[1]}\big) = -\mathcal{R}\big(\psi^2_{(4)[1]}\big) = 0, \tag{83}
$$

so that the renormalized value $\big(\psi^2_{(4)[1]}\big)_{\mathcal{R}} = \psi^2_{(4)[1]} + S_{\mathcal{R}}\big(\psi^2_{(4)[1]}\big)$ coincides with $\psi^2_{(4)[1]}$. For the 2-primitive $\psi^1_{(4)[2]}$, the first of (65) and (36) give

$$
\Delta\big(\psi^1_{(4)[2]}\big) = \psi^1_{(4)[2]} \otimes 1 + 1 \otimes \psi^1_{(4)[2]} + \frac{1}{2}\psi^1_{(1)[1]} \otimes \psi^1_{(3)[1]} - \frac{1}{2}\psi^1_{(3)[1]} \otimes \psi^1_{(1)[1]}, \tag{84}
$$

so that

$$
\big(\psi^1_{(4)[2]}\big)_{\mathcal{R}} = \psi^1_{(4)[2]} + S_{\mathcal{R}}\big(\psi^1_{(4)[2]}\big) + \frac{1}{2} S_{\mathcal{R}}\big(\psi^1_{(1)[1]}\big)\psi^1_{(3)[1]} - \frac{1}{2} S_{\mathcal{R}}\big(\psi^1_{(3)[1]}\big)\psi^1_{(1)[1]}. \tag{85}
$$

For the (non-trivial) twisted antipode we find

$$
S_{\mathcal{R}}\big(\psi^1_{(4)[2]}\big) = -\mathcal{R}\big(\psi^1_{(4)[2]}\big) + \frac{1}{2}\mathcal{R}\Big(\mathcal{R}\big(\psi^1_{(1)[1]}\big)\psi^1_{(3)[1]}\Big) - \frac{1}{2}\mathcal{R}\Big(\mathcal{R}\big(\psi^1_{(3)[1]}\big)\psi^1_{(1)[1]}\Big). \tag{86}
$$

Substituting above we get

$$
\big(\psi^1_{(4)[2]}\big)_{\mathcal{R}} = \frac{7}{96}\pi^4 + \mathcal{O}(\epsilon). \tag{87}
$$

Finally, for $\psi^{\stackrel{\bullet}{\bullet\wedge\bullet}}$, we use the coproduct given in (40) and, proceeding along the same lines, we find

$$\left(\psi^{\stackrel{\bullet}{\bullet\wedge\bullet}}\right)_{\mathcal{R}} = \frac{13}{96}\pi^4 - \frac{1}{24}\pi^2 a^2 + \mathcal{O}(\epsilon), \tag{88}$$

which is different, as mentioned above, from the finite $\psi^{\stackrel{\bullet}{\bullet\wedge\bullet}}$.

The remarkable pole structure of the ψ's observed above, persists for other, more realistic models as well. For example, we have repeated the above analysis for the heavy-quark model of [2]. We find that, for $n \leq 4$, the maximal pole order appearing is only $1/\epsilon$, with all ladder ψ's, except the first one, finite.

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