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HIGHER ORDER MEASURES, GENERALIZED QUANTUM MECHANICS AND HOPF ALGEBRAS

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We study Sorkin's proposal of a generalization of quantum mechanics and find that the theories proposed derive their probabilities from kth order polynomials in additive measures, in the same way that quantum mechanics uses a probability bilinear in the quantum amplitude and its complex conjugate. Two complementary approaches are presented, a C^* and a Hopf-algebraic one, illuminating both algebraic and geometric aspects of the problem.

Keywords: Quantum states; Hopf algebras; C^* algebras; generalized measures.

1. Introduction

In a series of papers,[9,10] Sorkin has put forward a view of quantum mechanics as a "quantum measure theory". His approach views the transition from classical to quantum mechanics as a generalization of the additivity properties of the classical measure function on a set of histories. This generalization has a natural extension, producing a whole family of theories, indexed by a positive integer k, each defined by a particular "sum rule" imposed on the measure function. Our purpose in this paper is to show that the various theories thus obtained can be characterized by the fact that the corresponding measure is a polynomial of degree k in primitive (i.e. additive) functionals.

The structure of the paper is as follows. In the rest of the Introduction we summarize earlier results by Sorkin. In Sec. 2, we approach the problem from an algebraic point of view. Section 3 complements the analysis in a geometrical spirit, using the language of Hopf algebras. Section 4, somewhat independent from the rest of the paper, sketches the relevance of these matters to theories designed to overcome the obstacles to locality imposed by Bell's inequalities. A concluding section suggests that experiments should be done to establish the k of nature and points to formal interconnections with an already existing concept of k-primitiveness in the literature.

We start with a description of the two-slit interference experiment, following closely the exposition in Refs. 8 and 9. Referring to the standard two-slit setup, we call H the set of all electron histories (worldlines) leaving the electron gun and arriving at the detector at specified time instants (to avoid technicalities we consider H to be measurable). We call A (B) the subset of H consisting of all histories in which the electron passes through slit a (b) (we ignore the possibility of the electron winding around both slits). Consider the four possible ways of blocking the two slits and denote by P_{ab}, P_a, P_b and $P_0 = 0$ the corresponding probabilities of arrival at the detector, the last one corresponding to both slits being blocked off. The idea now is to consider these probabilities as the values of a certain measure function μ defined on the set of subsets of H, e.g. $P_a = \mu(A)$. When mutually exclusive alternatives exist, as when both slits are open, the union of the corresponding (disjoint) subsets is to be taken, e.g. $P_{ab} = \mu(A \sqcup B)$ ($\sqcup$ denotes disjoint union). Physical theories are distinguished by their measures, for example, classical mechanics uses a "linear" measure μ_{cl}, satisfying the sum rule

$$I_2^{\mu_{\mathrm{cl}}}(A, B) \equiv \mu_{\mathrm{cl}}(A \sqcup B) - \mu_{\mathrm{cl}}(A) - \mu_{\mathrm{cl}}(B) = 0\,, \tag{1}$$

and hence fails to account for any interference. Quantum mechanics uses μ_{q}, satisfying $I_2^{\mu_{\mathrm{q}}}(A, B) \neq 0$, as is well known. Sorkin's observation was that in a three-slit experiment (with eight possibilities for blocking the slits), the probabilities predicted by quantum mechanics *do* satisfy the sum rule

$$\begin{aligned} I_3^{\mu_{\mathrm{q}}}(A, B, C) \equiv {} & \mu_{\mathrm{q}}(A \sqcup B \sqcup C) - \mu_{\mathrm{q}}(A \sqcup B) - \mu_{\mathrm{q}}(A \sqcup C) - \mu_{\mathrm{q}}(B \sqcup C) \\ & + \mu_{\mathrm{q}}(A) + \mu_{\mathrm{q}}(B) + \mu_{\mathrm{q}}(C) \\ = {} & 0\,, \end{aligned} \tag{2}$$

arguably a lesser known fact. It is easy to show that μ_{cl} also satisfies (2), as a result of (1). There is an obvious generalization to the k-slit experiment, involving the symmetric functional I_k^{μ}, given by

$$\begin{aligned} I_k^{\mu}(A_1, \ldots, A_k) \equiv {} & \mu(A_1 \sqcup \cdots \sqcup A_k) - \sum_i \mu(A_1 \sqcup \cdots \sqcup \hat{A}_i \sqcup \cdots \sqcup A_k) \\ & + \sum_{i<j} \mu(A_1 \sqcup \cdots \sqcup \hat{A}_i \sqcup \cdots \sqcup \hat{A}_j \sqcup \cdots \sqcup A_k) \cdots \\ & + (-1)^{k+1} \sum_i \mu(A_i)\,, \end{aligned} \tag{3}$$

where the hats denote omission and all A_i are mutually disjoint. These functionals satisfy the recursion relation

$$I^{\mu}_{k+1}(A_0, A_1, \ldots, A_k) = I^{\mu}_{k}(A_0 \sqcup A_1, A_2, \ldots, A_k) \\ - I^{\mu}_{k}(A_0, A_2, \ldots, A_k) - I^{\mu}_{k}(A_1, A_2, \ldots, A_k)\,, \qquad (4)$$

which implies that the sum rule $I^{\mu}_{k+1} = 0$ follows from $I^{\mu}_{k} = 0$. One may now contemplate a family of theories, indexed by a positive integer k, defined by the sum rule $I^{\mu}_{k+1} = 0$, with $I^{\mu}_{k} \neq 0$ for the corresponding measure. Classical mechanics is seen to be a $k = 1$ theory while quantum mechanics corresponds to $k = 2$.

The above formulas for I^{μ}_{k} need to be extended to the general case, i.e. when the arguments are possibly overlapping sets. For the $k = 2$ case, Sorkin gives the following equivalent forms

$$I^{\mu}_{2} = \mu(A \cup B) + \mu(A \cap B) - \mu(A \backslash B) - \mu(B \backslash A) \\ = \mu(A \triangle B) + \mu(A) + \mu(B) - 2\mu(A \backslash B) - 2\mu(B \backslash A)\,, \qquad (5)$$

derived by demanding bilinearity (the symbol $\backslash$ above denotes set-theoretic difference while $\triangle$ denotes symmetric difference).

2. The Formalism of Nonlinear Measures

2.1. *Preliminary considerations*

In the spirit of the functional theoretic formulation of the classical measure theory, we are now going to introduce an algebraic setup. The idea is to move from the language of sets to the language of functions, replacing the notion of measure by that of integral.

Let us consider a unital $\mathbb{Q}$-algebra A. We shall deal with certain nonlinear functionals

$$\mu\colon A \to \mathbb{C}$$

defined by a hierarchy of interesting algebraic relations.

For each $n \in \mathbb{N}$, let $\mathcal{M}_n(A)$ be the space of all maps μ satisfying

$$\mu(a_1 + \cdots + a_{n+1}) = \sum_{S} (-)^{n-|S|} \mu\left(\sum_{i \in S} a_i\right), \qquad (6)$$

where the $S \subset \{1, \ldots, n+1\}$ runs over all subsets satisfying $1 \leq |S| \leq n$.

It is easy to see that each $\mathcal{M}_n(A)$ is an A-bimodule, in a natural manner. The additive structure is trivial, while the left and right multiplications are given by

$$(x\mu y)(a) = \mu(yax)\,, \qquad x, y \in A\,.$$

Also, every $\mathcal{M}_n(A)$ allows multiplications by complex numbers (it is a complex vector space). Let us denote by $\Sigma_n(A)$ the space of multiadditive maps

$$\Phi\colon \overbrace{A \times \cdots \times A}^{n} \to \mathbb{C}$$

which are totally symmetric. The elements of $\Sigma_n(A)$ are naturally interpretable as *homogeneous polynomials* of order n over A.

2.2. *Quadratic measures*

We shall first analyze a special case of *quadratic* measures (corresponding to $n = 2$). As mentioned in the Introduction, this completely covers probability aspects of standard quantum mechanics. Because of the importance of this special case, we shall present all calculations independently of the general setting, which will be discussed in the next subsection.

Let us consider an arbitrary $\mu \in \mathcal{M}_2(A)$. The elements μ are characterized by the following identity

$$\begin{aligned}\mu(a+b+c) &= \mu(a+b)+\mu(a+c)+\mu(b+c)\\ &\quad -\mu(a)-\mu(b)-\mu(c)\,, \qquad \forall\ a,b,c\in A\,.\end{aligned} \tag{7}$$

As the first elementary consequence, it is worth observing that

$$\mu(0) = 0\,.$$

Furthermore, the group $\mathbb{Z}_2$ naturally acts on the space $\mathcal{M}_2(A)$. The action is induced by right multiplication by $-1 \in A$. It follows immediately that the space $\mathcal{M}_2(A)$ is naturally decomposed into a direct sum

$$\mathcal{M}_2(A) = \mathcal{M}_2^-(A) \oplus \mathcal{M}_2^+(A) \tag{8}$$

of "even" and "odd" subspaces:

$$\left.\begin{aligned}\mathcal{M}_2^-(A) &= \{\mu|\mu(a) = -\mu(-a)\}\,,\\ \mathcal{M}_2^+(A) &= \{\mu|\mu(a) = \mu(-a)\}\,,\end{aligned}\right. \qquad \forall\ a\in A\,. \tag{9}$$

Let us first analyze the odd part. As the following lemma shows, there is nothing very exciting about $\mathcal{M}_2^-(A)$.

Lemma 2.1. *The space $\mathcal{M}_2^-(A)$ is the set of $\mathbb{Q}$-linear maps $\mu: A \to \mathbb{C}$.*

Proof. It is obvious that all $\mathbb{Q}$-linear μ belong to $\mathcal{M}_2^-(A)$. Let us observe that $\mathbb{Q}$-linearity is equivalent to *additivity.* Therefore what remains is to prove that every $\mu \in \mathcal{M}_2^-(A)$ is additive. Indeed, replacing $c = -b$ in (7) and using the imparity assumption, we obtain

$$\mu(a+b)+\mu(a-b) = 2\mu(a)\,.$$

Interchanging a and b in the above identity, we obtain

$$\mu(a+b)-\mu(a-b) = 2\mu(b)\,.$$

Now summing the two equations we finally conclude

$$\mu(a+b) = \mu(a) + \mu(b)$$

which completes the proof. □

The space $\mathcal{M}_2^+(A)$ possesses a much more interesting structure. For a given $\mu \in \mathcal{M}_2^+(A)$ let us define a map Φ: $A \times A \to \mathbb{C}$ by

$$\Phi(a,b) = \frac{1}{4}(\mu(a+b) - \mu(a-b)) . \tag{10}$$

It follows immediately that

$$\Phi(a,b) = \Phi(b,a) , \tag{11}$$

i.e. the map Φ is symmetric.

Lemma 2.2. (i) *The map Φ is $\mathbb{Q}$-bilinear. In other words,*

$$\Phi(\lambda a + b, c) = \lambda\Phi(a,c) + \Phi(b,c) , \qquad \forall \;\; a,b,c \in A, \quad \lambda \in \mathbb{Q} . \tag{12}$$

(ii) *We can reconstruct μ from Φ by*

$$\mu(x) = \Phi(x,x) . \tag{13}$$

The correspondence $\mu \leftrightarrow \Phi$ is a natural isomorphism between the space $\Sigma_2(A)$ of symmetric bilinear functionals over A and the space of even quadratic measures $\mathcal{M}_2^+(A)$.

Proof. Let us observe that $\mathbb{Q}$-bilinearity is equivalent to biadditivity. Using (7) and performing elementary transformations we obtain

$$\begin{aligned}
\Phi(a,b+c) &= \frac{1}{4}(\mu(a+b+c) - \mu(a-b-c)) \\
&= \frac{1}{4}(\mu(a+b) + \mu(a+c) + \mu(b+c) - \mu(a-b) - \mu(a-c) - \mu(b+c)) \\
&= \Phi(a,b) + \Phi(a,c) ,
\end{aligned}$$

which proves (i). Now using the bilinearity property of Φ we find

$$\Phi(x,x) = 4\Phi(x/2,x/2) = \mu(x) - \mu(0) = \mu(x) .$$

Finally, it is straightforward to see that every $\Phi \in \Sigma_2(A)$ gives rise, via (13), to an even element $\mu \in \mathcal{M}_2(A)$. □

2.3. *Higher-order generalizations*

In this subsection we shall generalize the previous analysis for arbitrary degrees $n \in \mathbb{N}$. Let us introduce, in an algebraic analogy with Ref. 9, functionals

$$I_k^\mu(a_1, \ldots, a_k) = \sum_S (-)^{k-|S|} \mu\left(\sum_{i \in S} a_i\right), \tag{14}$$

where $k \geq 2$, the summation is over all nonempty subsets $S \subseteq \{1, \ldots, k\}$ and μ: $A \to \mathbb{C}$ is an arbitrary map. By definition, all the maps I_k are symmetric.

Lemma 2.3. *Let us assume that μ is arbitrary. We have*

$$I_{k+1}^\mu(b, c, a_2, \ldots, a_k) = I_k^\mu(b + c, a_2, \ldots, a_k) - I_k^\mu(b, a_2, \ldots, a_k) - I_k^\mu(c, a_2, \ldots, a_k)$$

for each $a_i, b, c \in A$ and $k \geq 2$.

Proof. This is just a straightforward combinatorial calculation, involving sums over different types of index subsets S: those that "contain" both b and c, subsets containing only symbol b or c, and those S excluding symbols b and c. □

It is easy to see that the following equivalences hold,

$$\mu \in \mathcal{M}_n(A) \Leftrightarrow I_{n+1}^\mu = 0, \qquad I_{n+1}^\mu = 0 \Leftrightarrow I_n^\mu \in \Sigma_n(A). \tag{15}$$

Taking into account the previous lemma, we conclude that $\mu \in \mathcal{M}_n(A)$ if and only if the functional I_n^μ is multiadditive (and hence $\mathbb{Q}$-multilinear). Hence, in this case we have $I_n^\mu \in \Sigma_n(A)$. Furthermore, we find

$$\mathcal{M}_{n-1}(A) \subseteq \mathcal{M}_n(A), \tag{16}$$

in other words, $\mathcal{M}_k(A)$ form a monotonically increasing family of A-modules.

From now on, let us assume that $\mu \in \mathcal{M}_n(A)$ and define a map Φ: $A^{\times n} \to \mathbb{C}$ by

$$\Phi(a_1, \ldots, a_n) = \frac{1}{2^n n!} \sum_z (-)^z \mu(z_1 a_1 + \cdots + z_n a_n), \tag{17}$$

where $z_i \in \{1, -1\}$ and $z = (z_1, \ldots, z_n)$.

Lemma 2.4. *The following identity holds*

$$I_n^\mu(a_1, \ldots, a_n) = n! \Phi(a_1, \ldots, a_n). \tag{18}$$

Proof. A direct calculation gives

$$\begin{aligned} 2^n I_n^\mu(a_1, \ldots, a_n) &= \sum_z (-)^z I_n^\mu(z_1 a_1, \ldots, z_n a_n) \\ &= \sum_{z,S} (-)^{z+n-|S|} \mu\left(\sum_{i \in S} z_i a_i\right) \\ &= \sum_z (-)^z \mu\left(\sum_{i=1}^n z_i a_i\right) = 2^n n! \Phi(a_1, \ldots, a_n) \end{aligned}$$

and hence (18) holds. We have applied the multilinearity property of I_n in the above calculation. □

Let us denote by $\Pi_n \colon \mathcal{M}_n(A) \to \Sigma_n(A)$ a linear map defined by $\Pi_n(\mu) = \Phi$. Using the previous lemma, and (16) we find

$$\ker(\Pi_n) = \mathcal{M}_{n-1}(A)\,. \tag{19}$$

The map Π_n is really a projection, and it admits a natural right section. Let us define $\iota_n \colon \Sigma_n(A) \to \mathcal{M}_n(A)$ by

$$\mu(x) = \Phi(\overbrace{x, \dots, x}^{n})\,, \qquad \mu = \iota_n(\Phi)\,. \tag{20}$$

Before going further, we have to verify that the image of ι_n is indeed within the space $\mathcal{M}_n(A)$. A direct calculation gives

$$\begin{aligned}
\sum_S (-)^{n-|S|} \mu\left(\sum_{i\in S} a_i\right) &= \sum_S (-)^{n-|S|} \Phi\left(\sum_{i\in S} a_i, \dots, \sum_{i\in S} a_i\right) \\
&= \sum_S (-)^{n-|S|} \sum_\alpha \Phi(a_{i_1}, \dots, a_{i_n}) \\
&= \sum_\alpha \Phi(a_{i_1}, \dots, a_{i_n}) = \mu(a_1 + \cdots + a_{n+1})\,,
\end{aligned}$$

where $\alpha = (i_1, \dots, i_n)$. The sumation is over subsets $S \subset \{1, \dots, n+1\}$ satisfying $1 \le |S| \le n$. The last equality is obtained as follows. Let us focus on an index term $(i_1, \dots, i_n)$ having exactly k different elements. The coefficient of this term is calculated by counting all the enveloping subsets S, with the corresponding signs. We arrive at

$$\begin{aligned}
\sum_{l=k}^{n} (-)^{n-l} \binom{n+1-k}{l-k} &= (-)^{n-k} \sum_{l=0}^{n-k} (-)^l \binom{n+1-k}{l} \\
&= -(-)^{n-k}(-)^{n-k+1} = 1\,.
\end{aligned}$$

Hence, $\mathrm{im}(\iota_n) \subseteq \mathcal{M}_n(A)$. It is easy to see that

$$\Pi_n \iota_n(\Phi) = \Phi\,, \qquad \forall\ \Phi \in \Sigma_n(A)\,. \tag{21}$$

Indeed, for $\mu = \iota_n(\Phi)$ we have

$$\begin{aligned}
\Pi_n(\mu)(a_1, \dots, a_n) &= \frac{1}{2^n n!} \sum_z (-)^z \mu(z_1 a_1 + \cdots + z_n a_n) \\
&= \frac{1}{2^n n!} \sum_z (-)^z \Phi\left(\sum_i z_i a_i, \dots, \sum_i z_i a_i\right) \\
&= \frac{1}{2^n n!} \sum_z (-)^z \sum_\alpha z_{i_1} \cdots z_{i_n} \Phi(a_{i_1}, \dots, a_{i_n}) = \Phi(a_1, \dots, a_n)\,.
\end{aligned}$$

The last equality is obtained by observing that only multi-indices $\alpha = (i_1, \ldots, i_n)$ that are *permutations* count, as other terms would cancel each other. The factor $2^n n!$ emerges as we sum over $\mathbb{Z}_2^n \times S_n$.

Summarizing our considerations we can now formulate

Proposition 2.1. *For every $n \geq 2$, there is a natural split short exact sequence*

$$0 \to \mathcal{M}_{n-1}(A) \hookrightarrow \mathcal{M}_n(A) \xrightarrow{\Pi_n} \Sigma_n(A) \to 0\,, \qquad \iota_n\colon \Sigma_n(A) \to \mathcal{M}_n(A) \tag{22}$$

which allows us to introduce a canonical decomposition

$$\mathcal{M}_n(A) \leftrightarrow \mathcal{M}_{n-1}(A) \oplus \Sigma_n(A)\,. \tag{23}$$

The elements of $\mathcal{M}_n(A)$ are nothing but polynomial functions of order (less than or equal to) n. In terms of the above identification, the elements of $\Sigma_n(A)$ correspond to homogeneous polynomials of order n.

3. Hopf Algebras and Generalized Measures

3.1. *Hopf algebras*

We give here a few basic definitions about Hopf algebras and some intuitive comments concerning their content. We keep the discussion informal, our basic aim being to point out the relevance of Hopf algebraic concepts to the problem at hand.

Restricted to the cocommutative case (we explain the term below), which is the one of interest here, the axioms for a Hopf algebra are just dual to those for a group. The duality is the one between points of the group manifold G and functions on the manifold and is formally expressed via an inner product,

$$\langle \cdot\,, \cdot \rangle\colon \quad \mathcal{A} \otimes G \to \mathbb{C}\,, \qquad f \otimes g \to \langle f, g \rangle \equiv f(g)\,, \tag{24}$$

extended by linearity to the group algebra. $\mathcal{A} \equiv C^\infty(G)$ is the (commutative) algebra of smooth complex valued functions on G while the last equation above simply states that the duality mentioned is by pointwise evaluation. The definition of a group involves the notions of a product $m\colon G \otimes G \to G$, an identity $e \in G$ and an inverse, which dualize, via the above inner product, to the notion of a *coproduct* Δ,

$$\Delta\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}\,, \qquad f \mapsto \Delta(f) \equiv \sum_i f^i_{(1)} \otimes f^i_{(2)} \equiv f_{(1)} \otimes f_{(2)}\,, \tag{25}$$

a *counit* ε,

$$\varepsilon\colon \mathcal{A} \to \mathbb{C}\,, \qquad f \mapsto \varepsilon(f)\,, \tag{26}$$

and a *coinverse* or *antipode* S,

$$S\colon \mathcal{A} \to \mathcal{A}\,, \qquad f \mapsto S(f)\,, \tag{27}$$

respectively. The defining relations are

$$\begin{aligned}\langle f, m(g \otimes g')\rangle = \langle f, gg'\rangle \equiv \langle f_{(1)} \otimes f_{(2)}\,, g \otimes g'\rangle &= \langle f_{(1)}\,, g\rangle\langle f_{(2)}\,, g'\rangle\,,\\ \varepsilon(f) &\equiv \langle f, e\rangle\,,\\ \langle S(f)\,, g\rangle &\equiv \langle f, g^{-1}\rangle\,,\end{aligned} \tag{28}$$

i.e. the Hopf algebraic operations are the adjoints, with respect to the above inner product, of those of a group.[a] When the group is Abelian (as in our case), the coproduct satisfies $f_{(1)} \otimes f_{(2)} = f_{(2)} \otimes f_{(1)}$ — in this case the Hopf algebra is called *cocommutative*. Notice that

$$\Delta(fh) = \Delta(f)\Delta(h)\,, \qquad \varepsilon(fh) = \varepsilon(f)\varepsilon(h)\,, \qquad S(fh) = S(h)S(f)\,, \tag{29}$$

where the product in $\mathcal{A} \otimes \mathcal{A}$ is defined by $(f \otimes h)(f' \otimes h') = ff' \otimes hh'$ (the primes distinguish functions here, they do not denote differentiation). Dual to the associativity of the group product is the *coassociativity* of the coproduct,

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta\,. \tag{30}$$

Then the notation Δ^k is unambiguous, since it does not matter to which tensor factor are the successive Δ's applied to — the resulting function of $k+1$ arguments will be denoted by $f_{(1)} \otimes \cdots \otimes f_{(k+1)}$ and it is invariant, in the cocommutative case, under exchange of any two tensor factors. Notice finally that dual to the property of the unit $ge = eg = g$ is the property of the counit

$$\varepsilon(f_{(1)})f_{(2)} = f_{(1)}\varepsilon(f_{(2)}) = f\,. \tag{31}$$

3.2. *Coderivatives*

One way of looking at the coproduct of a function is as an indefinite translation. Indeed, defining the right translation R_g on the group by $R_g(g') = g'g$, its pullback on functions $R_g^*(f) \equiv f_g$ is given by $f_g(g') = f(g'g) = f_{(1)}(g')f_{(2)}(g)$, which shows that $f_{(1)}(\cdot')f_{(2)}(g)$ is the right-translated f (by g), while $f_{(1)}(\cdot')f_{(2)}(\cdot)$, a function of two arguments, is the indefinitely translated f, with the second argument defining the translation and the first evaluating the translated function (one obtains a left version of the above exchanging the two tensor factors of the coproduct). With this in mind, one recognizes the operator $\mathcal{L}$: $\mathcal{A} \mapsto \mathcal{A} \otimes \mathcal{A}$, defined by

$$\mathcal{L}f = \Delta(f) - f \otimes 1\,, \tag{32}$$

as a (dualized) *indefinite discrete derivative* or *coderivative* for short,

$$\begin{aligned}(\mathcal{L}f)(g', g) &= \langle f_{(1)} \otimes f_{(2)} - f \otimes 1\,, g' \otimes g\rangle\\ &= f(g'g) - f(g')\,.\end{aligned} \tag{33}$$

[a]The above, although suitable for our purposes, is not the standard definition of a Hopf algebra. The latter can be consulted in, e.g. Ref. 11.

When g is close to the identity, $g = e + X + \cdots$, with X in the Lie algebra of the group, $(\mathcal{L}f)(\cdot', g)$ is (proportional to) the derivative of f along the left-invariant vector field corresponding to X. One may define higher order coderivatives $\mathcal{L}^k f$, with the understanding that the successive applications of $\mathcal{L}$ are to be taken at the leftmost tensor factor,

$$\mathcal{L}^k f \equiv (\mathcal{L} \otimes \mathrm{id}) \circ \mathcal{L}^{k-1} f\,, \qquad k = 2, 3, \ldots\,, \tag{34}$$

so that, for example,

$$\begin{aligned} \mathcal{L}^2 f &\equiv (\mathcal{L} \otimes \mathrm{id}) \circ \mathcal{L} f \\ &= (\mathcal{L} \otimes \mathrm{id})(f_{(1)} \otimes f_{(2)} - f \otimes 1) \\ &= f_{(1)} \otimes f_{(2)} \otimes f_{(3)} - f_{(1)} \otimes 1 \otimes f_{(2)} - f_{(1)} \otimes f_{(2)} \otimes 1 + f \otimes 1 \otimes 1\,. \end{aligned} \tag{35}$$

Of particular interest to us will be the evaluation of the above kth order coderivative at the identity of the group, $(\mathcal{L}^k f)(e, \cdot, \ldots) \equiv (\mathcal{L}^k f)(e)$, e.g.

$$\begin{aligned} (\mathcal{L}f)(e) &= f - \varepsilon(f)1\,, \\ (\mathcal{L}^2 f)(e) &= f_{(1)} \otimes f_{(2)} - f \otimes 1 - 1 \otimes f + \varepsilon(f) 1 \otimes 1\,, \end{aligned} \tag{36}$$

where (31) was used. We are now ready to introduce the basic notion of k-*primitiveness*

> **D1** *A function f will be called k-*primitive *if all its coderivatives at the identity $(\mathcal{L}^r f)(e)$, $r > k$ are equal to zero, while $(\mathcal{L}^k f)(e)$ is not.*

3.3. *Generalized quantum mechanics and k-primitiveness*

3.3.1. *Group structure on the set of histories*

Consider the set of histories H associated to some given experiment, taken as a measurable set for simplicity. For a subset A of H, let χ_A be the *characteristic function* of A, defined by $\chi_A(x) = 1$ if $x \in A$, $\chi_A(x) = 0$ if $x \in H \backslash A$. It is clear that one may deal with the subsets of H in terms of their characteristic functions, as we do in the following. Denote by G the set of all simple functions[b] on H, i.e. a typical element g of G is of the form $g = \lambda_1 \chi_{A_1} + \lambda_2 \chi_{A_2} + \cdots$, where the A_i are measurable subsets of H and $\lambda_i \in \mathbb{C}$. We may turn G into an Abelian group defining the group law by addition. Then for the identity e we have $e = \chi_\emptyset = 0$ and the inverse of g is $-g$.

Just like in the Introduction, a physical theory derives its probabilities from a measure function μ, defined now on G, e.g. $P_a = \mu(\chi_A)$ in the two-slit experiment. When mutually exclusive alternatives exist, the *sum of the characteristic functions of the corresponding subsets* is to be taken. Notice that, in terms of the subsets themselves, this corresponds to disjoint union, in accordance with the operation

[b]These are all linear combinations of characteristic functions of measurable subsets of H.

used in Refs. 8 and 9. The important point is that simply by extending this definition (i.e. addition of the characteristic functions) to non-disjoint subsets we recover the rather complicated interference term (5) and its generalizations, as we now show. Indeed, consider a quadratic functional μ^2, with μ additive, evaluated on two overlapping subsets A and B — the resulting interference term is

$$\begin{aligned} I_2^{\mu^2} &= \mu(\chi_A + \chi_B)^2 - \mu(\chi_A)^2 - \mu(\chi_B)^2 \\ &= 2\mu(\chi_A)\mu(\chi_B) \\ &= 2(\mu(\chi_{A\setminus B})\mu(\chi_{B\setminus A}) + \mu(\chi_{A\setminus B})\mu(\chi_{A\cap B}) \\ &\quad + \mu(\chi_{A\cap B})\mu(\chi_{B\setminus A}) + \mu(\chi_{A\cap B})^2)\,, \end{aligned} \tag{37}$$

where, in the last step, we wrote $\chi_A = \chi_{A\setminus B} + \chi_{A\cap B}$ and similarly for χ_B. On the other hand, the first, for example, of (5) becomes

$$I_2^{\mu^2} = \mu(\chi_{A\cup B})^2 + \mu(\chi_{A\cap B})^2 - \mu(\chi_{A\setminus B})^2 - \mu(\chi_{B\setminus A})^2\,. \tag{38}$$

Substituting $\chi_{A\cup B} = \chi_{A\setminus B} + \chi_{B\setminus A} + \chi_{A\cap B}$ and expanding one recovers the right-hand side of (37).

3.3.2. *k-primitive functions on G*

We focus now on the commutative and cocommutative Hopf algebra $\mathcal{A} \equiv C^\infty(G)$ of smooth functions on G. Among its elements are the quantum measures μ we have been considering so far. The fact that $\mu(\emptyset) = 0$ translates, in the Hopf algebraic language of this section, into the statement that the counit of all measures vanishes, $\mu(e) = \varepsilon(\mu) = 0$. The linearity of the classical measure, Eq. (1), becomes here the statement that $\mu_{\rm cl}(\chi_A + \chi_B) = \mu_{\rm cl}(\chi_A) + \mu_{\rm cl}(\chi_B)$, which is easily seen to dualize to

$$0 = (\mathcal{L}^2\mu)(e) = \mu_{{\rm cl}\,(1)} \otimes \mu_{{\rm cl}\,(2)} - \mu_{\rm cl} \otimes 1 - 1 \otimes \mu_{\rm cl} + \varepsilon(\mu_{\rm cl}) 1 \otimes 1\,, \tag{39}$$

the last term being zero. Hence, according to (D1), $\mu_{\rm cl}$ is a one-primitive element of $\mathcal{A}$. More generally, we have the following:

Lemma 3.1. *The symmetric functionals I_k^μ, defined in Eq.* (3), *coincide with the kth order coderivatives $(\mathcal{L}^k\mu)(e)$ of Eq.* (34).

We omit the straightforward inductive proof. We may now state the main result of this section:

Proposition 3.1. *In the algebra $\mathcal{A}$ of functions on G, every k-primitive element is a kth degree polynomial in one-primitive elements.*

Proof. G, being an infinite-dimensional Abelian Lie group, admits one-primitive normal coordinates $\{\xi_i\}_{i\in\Lambda}$, where Λ is an infinite index set. Any element of $\mathcal{A}$, in particular a k-primitive measure μ, is a function of the ξ_i, $\mu = \mu(\xi_i)$. From the vanishing of $(\mathcal{L}^{k+1}\mu)(e)$, with $(\mathcal{L}^k\mu)(e) \neq 0$, one may infer, by evaluating on arguments infinitesimaly close to the identity of G, that $(X_{i_1}\cdots X_{i_{k+1}})(\mu)(e) = 0$, for all X_i in the Lie algebra of G, while $(X_{i_1}\cdots X_{i_k})(\mu)(e) \neq 0$ for at least one index set. Given that G is Abelian and the ξ_i are normal coordinates, one may choose the X_i to correspond to partial derivatives w.r.t. the ξ_i, $X_i = \frac{\partial}{\partial\xi_i}$. Then, the above property of μ implies that $\frac{\partial^{k+1}\mu}{\partial\xi_{j_1}\cdots\partial\xi_{j_{k+1}}}(e) = 0$, for all j_i, while at least one kth-order partial derivative is nonzero at the identity. The proposition then follows from elementary calculus. □

The same conclusion can be reached by establishing that $\mathcal{A}$ is a cocommutative graded connected Hopf algebra and hence, by applying the Milnor–Moore theorem,[7] isomorphic to the universal enveloping algebra of its subalgebra of one-primitive elements. We point out that in Ref. 8, it has been observed that if μ is the kth power of a "linear" functional then $I_r^\mu = 0$, for $r > k$.

4. A C^*-Algebraic Formulation

We are going to touch upon some interesting issues related to a C^*-algebraic formulation of the algebraic setup of Sec. 2. A special emphasis will be given to possible relationships between the introduced formalism, the theory of contextual hidden variables,[5] and a corresponding non-Kolmogorovian probability framework as a way of overcoming obstacles to locality, given by Bell's inequalities. In order to keep this section reasonably short, we will only sketch basic ideas, and leave detailed presentations with proofs for another article.

We will assume here that A is a C^*-algebra. By definition,[1] this means that A is a Banach algebra, equipped with a *-structure (antilinear and antimultiplicative involution), so that

$$|aa^*| = |a|^2\,, \qquad \forall\; a \in A\,.$$

A remarkable property of C^*-algebras is that the norm is uniquely fixed by the algebra structure. In other words, for a given *-algebra A, there is at most one C^*-algebraic norm. In such a way C^*-algebras form a full subcategory of complex *-algebras.

The algebra A is called unital if there is a (necessarily unique) unit element $1 \in A$. We will deal with unital algebras only.

From the point of view of our considerations, we can think of A as consisting of *physical observables*.[c] Two special cases are the most interesting here:

- Classical case — A is a commutative algebra, generated by certain functions on the system's phase space Γ. For example, we can assume $A = L^\infty(\Gamma)$. That is, A is the algebra of (classes of) essentially bounded measurable functions on Γ.

[c]More precisely, hermitian elements of A are viewed as physical observables.

- Quantum case — A is a noncommutative algebra, generated by operators acting in the Hilbert state space H. For example, we can assume $A = \mathbb{B}(H)$. In other words, A is the algebra of all bounded operators acting in H.

However, all our considerations apply to general C^*-algebras. Let us begin by recalling the concept of a *state*. This is any functional $\rho: A \to \mathbb{C}$ satisfying

$$\rho(a^*a) \geq 0\,, \qquad \forall\ a \in A\,,$$

$$\rho(1) = 1\,.$$

In other words, a state is a positive and normalized functional on A. It is easy to see that the set of all states on A is convex, and compact in the *-weak topology of the dual space A^*. According to Krein–Millman theorem, $\mathbb{S}(A)$ is the closure of the convex hull of its extremal elements. These extremal elements are called *pure states*.

The theory of states generalizes the classical probability theory, to the level of noncommutative (quantum) spaces. Indeed, if A is commutative, then according to the classical Gelfand–Naimark theorem, we have a natural identification

$$A \leftrightarrow C(X)\,,$$

where X is a compact topological space — the spectrum of A (the set of all characters $\kappa: A \to \mathbb{C}$, equipped with the *-weak topology of the dual space A^*). In this commutative case, states on A correspond, according to the classical Riesz representation theorem, to probability measures on X. The correspondence is given by the Lebesgue integral.

Taking into account the considerations of Sec. 2, it is natural to formulate:

Definition 4.1. A *generalized*, *order-n*, *state* on A is a map $\rho \in \Sigma_n(A)$ satisfying

$$\rho(a^*a) \geq 0\,, \qquad \forall\ a \in A\,,$$

$$\rho(1) = 1\,.$$

Let us denote by $\mathbb{S}_n(A)$ the set of such order-n states on A. It is easy to see that $\mathbb{S}_n(A)$ is convex, and can be equipped with a natural *-weak topology, converting it into a compact topological space. Applying the Krein–Millman theorem, it follows that $\mathbb{S}_n(A)$ is the closure of the convex hull of its extremal elements.

Definition 4.2. The extremal elements of $\mathbb{S}_n(A)$ are called *pure* (*order-n*) *states* on the algebra A.

Let us assume that A is generated by its projectors (hermitian idempotents $p = p^* = p^2$). Such elements correspond to elementary yes/no situations, and can be viewed as the simplest possible physical observables. We can also identify projectors with *events*. In the classical case projectors correspond to the appropriate subsets of the phase space.

So, given a projector $p \in A$ and a higher-order state $\rho: A \to \mathbb{C}$, we want to interpret the number $\rho(p) \in [0,1]$ as the *probability of the event p in the state ρ*, just as in the standard case (otherwise, it would not be much of a sense to formulate the above definitions).

However, higher-order states contain an additional obstacle for such an interpretation. Let us consider two events $p, q \in A$ that are realizable simultaneously (within the same experimental context, this implies that $pq = qp$). Let us also assume that p and q are mutually exclusive. This means that $pq = qp = 0$ so we have orthogonal projectors. If our higher-order state ρ represents something really meaningful, then we must have

$$\rho(p+q) = \rho(p) + \rho(q)\,. \tag{40}$$

The above condition is automatically fulfilled for standard states (due to linearity). For all higher-order states, the condition is actually a condition for p and q. In particular, if we put $q = 1 - p$ we get a nontrivial algebraic condition on a single event p. In other words, *not all events are allowed.* Of course, it might happen that for a given ρ there are no nontrivial projectors p satisfying the consistency condition. In this case, the state ρ is basically useless, from the point of view of the statistical interpretation of its values on projectors. On the other hand, the states that always satisfy the consistency condition (for every orthogonal events p and q) are, in all non-perverted scenarios, just the standard linear states. This follows from the generalized Gleason theorem by Maeda.[6]

Therefore, in order for a higher-order state ρ to be *reasonable*, it should have sufficiently many "good" projectors p. This motivates our next definition.

Definition 4.3. Let us consider a higher-order state $\rho \in \mathbb{S}_n(A)$. A projector $p \in A$ is called ρ-*compatible* if

$$\rho(p) + \rho(1-p) = 1. \tag{41}$$

The state ρ is called A-*compatible* if the set of all ρ-compatible projectors generates the whole C^*-algebra A. Finally, for a given A-compatible state ρ, a unital C^*-subalgebra $B \subseteq A$ is called ρ-*compatible*, if (40) holds for all mutually orthogonal projectors from B.

Let us assume that ρ is an arbitrary A-compatible higher-order state. Then it gives rise to a nice short exact sequence of C^*-algebras:

$$0 \to \mathcal{K} \hookrightarrow \hat{A} \xrightarrow{\pi} A \to 0\,. \tag{42}$$

Here $\hat{A}$ is the free C^* algebra generated by all ρ-compatible subalgebras B of A. The map $\pi: \hat{A} \to A$ is the natural projection homomorphism and $\mathcal{K} = \ker(\pi)$.

In a special case when A is commutative, the above exact sequence is very similar to a class of contextual subquantum extensions considered in Ref. 5. Indeed, we can write $A = C(\Omega)$ where Ω is the spectrum of A (interpreted here as the subquantum space of the system) and our extension becomes:

$$0 \to \mathrm{com}(A) \hookrightarrow \hat{A} \xrightarrow{\pi} C(\Omega) \to 0\,. \tag{43}$$

The ρ-compatible subalgebras B correspond to allowed *measurement contexts* in Ω. We see that the probability theory on Ω is a non-Kolmogorovian one, in the case of higher-order states ρ: the additivity of the measure holds only within the measurement contexts. The noncommutative algebra $\hat{A}$ corresponds to the *full subquantum algebra*. The kernel of π is simply the commutant of A. Such models overcome obstacles to locality given by Bell's inequalities, because they are based on the appropriate non-Kolmogorovian statistics. The composite systems are simply described by taking tensor products of the introduced extensions.

5. Conclusions and Final Remarks

We have studied Sorkin's hierarchy of generalizations of quantum mechanics and found that the kth-order generalized measures are necessarily kth degree polynomials in one-primitive functionals, in the same sense that standard quantum mechanics derives its probabilities from a bilinear expression in a one-primitive (i.e. additive) quantum amplitude and its (also one-primitive) complex conjugate. The question of how is positivity to be attained in a, for example, cubic theory is still open. On the other hand, one may envisage a $k = 4$ theory as a small quartic correction to the standard quantum mechanical probability, showing up as a small deviation of I_4^μ from zero in a four-slit experiment. What we find remarkable is the immediacy with which the sum rules $I_k^\mu = 0$ connect to a simple k-slit experiment, a point that might be worth bringing to the attention of our experimental colleagues.

On a more formal level, a very important subject is the study of interrelations between states and representations of C^*-algebras generated by physical observables. According to the GNS construction,[1] there is a natural one-to-one correspondence

$$\{\text{Standard states } \rho \text{ on } A\} \Leftrightarrow \{\text{equivalence classes of cyclic representations of } A\}\,.$$

In terms of this correspondence, pure states translate into irreducible representations. The generalization of the GNS construction for the higher-order states introduced in this paper is a subject for further research.

It is worth mentioning that the extensions of commutative C^*-algebras of Sec. 4 by noncommutative ones play a central role in algebraic K-theory and noncommutative geometry.[2,12] For example, noncommutative extensions similar to (43) can be used to build a K-homology theory for metrizable compact topological spaces Ω. Another interesting question, suggested by the referee, would be to study possible generalizations of the Riesz representation theorem to the higher-order formalism studied in this paper.

We end this paper by pointing out that a concept of k-primitiveness has appeared recently in the study of the Hopf algebra structure in the process of renormalization in quantum field theory (see Ref. 3). A second, esssentially equivalent, definition was given and the concept was further analyzed in Ref. 4. In those works it refers to the much more complicated case of the Hopf algebra of rooted trees but these earlier definitions can be shown to be identical with the one presented here, although, we feel, the latter clarifies the underlying geometrical content. We plan to further elucidating these interconnections in an upcoming publication.

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