FTUV/98-62, IFIC/98-63, DAMTP 1998-141

Higher order BRST and anti–BRST operators and cohomology for compact Lie algebras

C. Chryssomalakos[1], J. A. de Azcárraga[1], A. J. Macfarlane[2] and J. C. Pérez Bueno[1*]

1 *Departamento de Física Teórica, Universidad de Valencia and IFIC, Centro Mixto Universidad de Valencia–CSIC E–46100 Burjassot (Valencia), Spain*

2 *Department of Applied Mathematics and Theoretical Physics, Silver St., Cambridge, CB3 9EW, UK*

Abstract

After defining cohomologically higher order BRST and anti-BRST operators for a compact simple algebra $\mathcal{G}$, the associated higher order Laplacians are introduced and the corresponding supersymmetry algebra Σ is analysed. These operators act on the states generated by a set of fermionic ghost fields transforming under the adjoint representation. In contrast with the standard case, for which the Laplacian is given by the quadratic Casimir, the higher order Laplacians W are not in general given completely in terms of the Casimir-Racah operators, and may involve the ghost number operator. The higher order version of the Hodge decomposition is exhibited. The example of $su(3)$ is worked out in detail, including the expression of its higher order Laplacian W.

*e-mail addresses: chryss@lie.ific.uv.es, azcarrag@lie1.ific.uv.es, a.j.macfarlane@damtp.cam.ac.uk, pbueno@lie.ific.uv.es

1 Introduction

BRST symmetry [1, 2], or 'quantum gauge invariance', has played an important role in quantisation of non-abelian gauge theories. The nilpotency of the operator $\mathcal{Q}$ generating the global BRST symmetry implies that the renormalisation of gauge theories involves cohomological aspects: the physical content of the theory belongs to the kernel of $\mathcal{Q}$, the physical (BRST invariant) states being defined by BRST–cocycles modulo BRST–trivial ones (coboundaries). The inclusion of the BRST symmetry in the Batalin–Vilkovisky antibracket–antifield formalism (see [3, 4] for further references), itself of a rich geometrical structure [3, 4, 5, 6, 7, 8, 9], and where the antifields are the sources of the BRST transformations, has made of BRST quantisation the most powerful method for quantising systems possessing gauge symmetries. In particular, it is indispensable for understanding the general structure of string amplitudes. It is thus interesting to explore its possible generalisations and their cohomological structure.

An essential ingredient of $\mathcal{Q}$ is what we shall denote here the BRST–*operator*

$$s = -\frac{1}{2} C_{ij}{}^{k} c^i c^j \frac{\partial}{\partial c^k} \quad , \quad i = 1, \ldots, r = \dim \mathcal{G} \quad , \tag{1.1}$$

where the c^i are anticommuting Grassmann (or *ghost*) variables transforming under the adjoint representation of the (compact semisimple) Lie group G of Lie algebra $\mathcal{G}$. In Yang Mills theories the c's correspond to the ghost fields, and (1.1) above is just part of the generator of the BRST transformations for the gauge group G. This paper is devoted to the generalisations of (1.1) and its associated *anti*–BRST *operator* $\overline{s}$ [10, 11, 12, 13, 14, 15, 16].

Using Euclidean metric to raise and lower indices[1] $\overline{s}$ is given by

$$\overline{s} = \frac{1}{2} C_{ij}{}^{k} c_k \frac{\partial}{\partial c_i} \frac{\partial}{\partial c_j} \quad . \tag{1.2}$$

The s ($\overline{s}$) operator increases (decreases) the ghost number by one. The BRST and anti–BRST operators may be used to construct a Laplacian [17, 12, 13, 18, 14, 16], $\Delta = \overline{s}s + s\overline{s}$; clearly, Δ does not change the ghost number. It turns out (see [12, 13, 14]) that this operator is given by the (second order) Casimir operator of $\mathcal{G}$.

A few years ago, van Holten [14] discussed the BRST complex, generated by the s_ρ and $\overline{s}_\rho$ operators,

$$s_\rho = c^i \rho(X_i) - \frac{1}{2} C_{ij}{}^{k} c^i c^j \frac{\partial}{\partial c^k} \ , \qquad \overline{s}_\rho = -\rho(X_i) \frac{\partial}{\partial c_i} + \frac{1}{2} C_{ij}{}^{k} c_k \frac{\partial}{\partial c_i} \frac{\partial}{\partial c_j} \ , \tag{1.3}$$

in connection with the cohomology of compact semisimple Lie algebras. They act on generic states of ghost number q of the form

$$\psi = \frac{1}{q!} \psi^A_{i_1 \ldots i_q} c^{i_1} \ldots c^{i_q} \otimes e_A \quad . \tag{1.4}$$

[1] The Killing tensor k_{ij} is proportional to δ_{ij} since $\mathcal{G}$ is compact and semisimple.

The operators in (1.3) differ from those in (1.1), (1.2) by the inclusion of the ρ term, where ${\rho^A}_B$ is a representation of the Lie algebra $\mathcal{G}$ on a vector space V with basis $\{e_A\}$, $A = 1, \ldots, \dim V$. However (see the *Remark* in section 2), only the trivial representation case is interesting. For ρ=0 the generic states have the form

$$\psi = \frac{1}{q!} \psi_{i_1 \ldots i_q} c^{i_1} \ldots c^{i_q} \quad , \tag{1.5}$$

and we shall consider mostly this case. The operators s, $\overline{s}$ and the Laplacian Δ may be used to define s–closed, $\overline{s}$–closed (coclosed) and harmonic states. A state ψ (eqn. (1.5)) is called s–closed, $\overline{s}$–closed or harmonic if $s\psi = 0$, $\overline{s}\psi = 0$ or $\Delta\psi = 0$ respectively. In this way, and using the nilpotency of s and $\overline{s}$, one may introduce a Hodge decomposition for (1.5) as a sum of an s–closed, a $\overline{s}$–closed and a harmonic state. The interesting fact is that, using the above Euclidean metric on $\mathcal{G}$, one may introduce a positive scalar product among states ψ', ψ, of ghost numbers q', q by

$$\langle \psi', \psi \rangle := \frac{1}{q!} \psi'_{j_1 \ldots j_q} \psi^{j_1 \ldots j_q} \delta_{q'q} \quad . \tag{1.6}$$

Using the Hodge $*$ operator for δ_{ij} it follows that $s = (-1)^{r(q+1)} * \overline{s} *$ (on states of ghost number q), and that s and $\overline{s}$ are also adjoint to each other with respect to the scalar product (1.6). As a result, there is a complete analogy between the harmonic analysis of forms, in which d and $\delta = (-1)^{r(q+1)+1} * d *$ are adjoint to each other, and the Hodge-like decomposition of states ψ for the operators s, $\overline{s}$ [14] (see also section 2). This follows from the fact that, due to their anticommuting character, the ghost variables c may be identified [19] with (say) the left-invariant one–forms on the group manifold G, so that the action of s on c determines the Maurer-Cartan (MC) equations.

The nilpotency of the BRST operators (1.1) or (1.3) results from the Jacobi identity satisfied by the structure constants ${C_{ij}}^k$ of $\mathcal{G}$. This identity can also be viewed as a three-cocycle condition on the fully antisymmetric C_{ijk}, which define a non-trivial three-cocycle for any semi-simple $\mathcal{G}$. This observation indicates the existence of a generalization by using the higher order cocycles for $\mathcal{G}$. The cohomology ring of all compact simple Lie algebras of rank l (for simplicity we shall assume G simple henceforth) is generated by l (classes of) non-trivial primitive cocycles, associated with the l invariant, symmetric primitive polynomials of order m_s, ($s = 1, \ldots, l$) which, in turn, define the l Casimir-Racah operators ([20, 21, 22]; see also [23] and references therein). The different integers m_s depend (for $s \neq$1) on the specific simple algebra considered. It has been shown in [24] that, associated to each cocycle of order $2m_s - 1$ there exists a higher order BRST operator s_{2m_s-2} carrying ghost number $2m_s - 3$, defined by the coordinates $\Omega^j_{i_1 \ldots i_{2m_s-2}}$ of the $(2m_s - 1)$-cocycle on $\mathcal{G}$ (we shall also give in (3.17) the corresponding operator $s_{\rho\,(2m_s-2)}$ for the $\rho \neq 0$ case). The $\Omega^j_{i_1 \ldots i_{2m_s-2}}$ may also be understood as being the (fully antisymmetric) higher order structure constants of a higher $(2m_s-2)$ order algebra [24], for which the multibrackets have $(2m_s-2)$ entries. The standard (lowest, s=1) case corresponds to $m_1 = 2\ \forall\ \mathcal{G}$ (the Cartan-Killing metric), to the three-cocycle C_{ijk} and to the ordinary Lie algebra bracket. The $(2m_s-2)$-brackets of these higher order algebras satisfy a generalized Jacobi identity which again follows from the fact that the higher order structure constants define $(2m_s-1)$-cocycles for the Lie algebra cohomology. These $(2m_s-2)$-algebras constitute a particular example (in which only one coderivation survives) of the strongly homotopy algebras [25], which have recently appeared in different physical theories which share common cohomological aspects, as in closed string field theory [26, 27], the higher order generalizations of the antibracket [28, 29] and the Batalin–Vilkovisky complex (see [30] and references therein). Higher order structure constants satisfying generalized Jacobi identities of the types considered in [24] (see (3.11) below) and [31] have also appeared in a natural way in the extended master equation in the presence of higher order conservation laws [9].

In section 3 of this paper we introduce, together with the l general BRST operators for a simple Lie algebra, the corresponding l anti–BRST operators $\overline{s}_{2m_s-2}$ and their associated higher order Laplacians. We show there that harmonic analysis may be carried out in general (the standard case in [14] corresponds to $m_s=m_1=2$), although the Laplacians do not in general correspond to the Casimir-Racah operators. Nevertheless, we shall show that s_{2m_s-2} and $\overline{s}_{2m_s-2}$ are related to each other by means of the Hodge $*$ operator, and that they are also adjoint of each other. After showing that the different higher order BRST, anti-BRST and Laplacian operators generate, for each value $s=1,\ldots,l$, a supersymmetry algebra Σ_{m_s}, we discuss its representations. The example of $\mathcal{G}$=$su(3)$ is studied in full in section 4 where we construct the general $su(3)$ states and show the $su(3)$ representations contained in the $\Sigma_{m_s}=\Sigma_2,\Sigma_4$ irreducible multiplets.

2 The standard BRST complex and harmonic states

Let $\mathcal{G}$ be defined by

$$[X_i,X_j]=C_{ij}{}^{k}X_k \quad , \quad i,j,k=1,\ldots,r\equiv\dim\mathcal{G} \quad , \tag{2.1}$$

where $\{X_i\}_{i=1}^{r}$ is a basis of $\mathcal{G}$. For instance, we may think of $\{X_i\}$ as a basis for the left invariant (LI) vector fields $X_i^L(g)\equiv X_i(g)$ on the group manifold G ($X_i(g)\in\mathfrak{X}^L(G)$).

Let V be a vector space. In the Chevalley-Eilenberg formulation (CE) [32] of the Lie algebra cohomology, the space of q–dimensional cochains $C^q(\mathcal{G},V)$ is spanned by the V–valued skew–symmetric mappings

$$\psi:\mathcal{G}\wedge\overset{q}{\cdots}\wedge\mathcal{G}\to V \quad , \quad \psi(g)=\frac{1}{q!}\psi^A_{i_1\ldots i_q}\omega^{i_1}(g)\wedge\ldots\wedge\omega^{i_q}(g)\otimes e_A \quad , \tag{2.2}$$

where the $\{\omega^i(g)\}$ form a basis of $\mathcal{G}^*$ (LI one-forms on G), dual to the basis $\{X_i\}$ of LI vector fields on G, and the index $A=1,\ldots,\dim V$ labels the components in V. Let ρ be a representation of $\mathcal{G}$ on V ($\rho:\mathcal{G}\to\mathrm{End}(V)$). The action of the Lie algebra coboundary operator s_ρ, ${s_\rho}^2=0$, on the q–cochains ψ^A (1.4) is given by

Definition 2.1 (*Coboundary operator*)
The coboundary operator $s_\rho : C^q(\mathcal{G},V) \to C^{q+1}(\mathcal{G},V)$ is defined by

$$
\begin{aligned}
(s_\rho\psi)^A(X_1,...,X_{q+1}) &:= \sum_{i=1}^{q+1}(-1)^{i+1}\rho(X_i)^A_{.B}\,(\psi^B(X_1,...,\hat{X}_i,...,X_{q+1})) \\
&+ \sum_{\substack{j,k=1\\ j<k}}^{q+1}(-1)^{j+k}\psi^A([X_j,X_k],\ X_1,...,\hat{X}_j,...,\hat{X}_k,...,X_{q+1}) \quad .
\end{aligned} \tag{2.3}
$$

The space of q–*cocycles* $Z^q_\rho(\mathcal{G},V)$ (*i.e.* $\mathrm{Ker}\, s$) modulo the q–coboundaries $B^q_\rho(\mathcal{G},V)$ (*i.e.* $\mathrm{Im}\, s$) defines the q–th Lie algebra cohomology group $H^q_\rho(\mathcal{G},V)$.

Since we are assuming $\mathcal{G}$ semisimple, Whitehead's lemma states that, for ρ non-trivial,

$$
H^q_\rho(\mathcal{G},V) = 0 \quad , \quad \forall q \geq 0 \quad , \tag{2.4}
$$

and we can restrict ourselves to $\rho = 0$ cohomology for which the action of s_ρ reduces to the second term in the r.h.s. of eqn. (2.3).

For the trivial representation, s acts on ψ (1.5) in the same manner as the exterior derivative d acts on LI forms. It is then clear that we may replace the $\{\omega^i(g)\}$ by the ghost variables $\{c^i\}$,

$$
c^i c^j = -c^j c^i \quad (\{c^i, c^j\} = 0\,, \quad \{c^i, \frac{\partial}{\partial c^j}\} = \delta^i_j) \quad , \quad i,j = 1,\ldots,r \quad , \tag{2.5}
$$

and the space of q–cochains by polynomials of (ghost) number $q \leq r$. The BRST operator (1.1) $s = s_2$ (the subindex 2 is added for convenience; its meaning will become clear in section 3) may be taken as the coboundary operator for the (ρ=0) Lie algebra cohomology [19]. Indeed, the relations

$$
s_2 c^k = -\frac{1}{2} C_{ij}{}^k c^i c^j \qquad (\text{or } s_2 c = -\frac{1}{2}[c,c]\ ,\ c = c^i \rho(X_i)\,) \quad , \tag{2.6}
$$

reproduce the MC equations. As a result, the Lie algebra cohomology may be equivalently formulated in terms of skew-symmetric tensors on $\mathcal{G}$, LI forms on G, or polynomials in ghost space (see *e.g.* [33]).

In the sequel we shall introduce the Grassmann variables π_i to refer to the 'partial derivative' $\partial/\partial c^i$, appropriate for using the 'ghost representation' for the cochains/states. These two sets of variables (c^i, π_j) span a Clifford–like algebra[2] defined by

$$
\{c_i, \pi_j\} = \delta_{ij} \quad , \quad \{c_i, c_j\} = 0 = \{\pi_i, \pi_j\} \quad . \tag{2.7}
$$

[2] The algebra (2.7) can be represented by a Clifford algebra (see *e.g.* [14]). Namely, if we define $c_i = \frac{1}{2}(\gamma_i - i\gamma_{i+r})$, $\pi_i = \frac{1}{2}(\gamma_i + i\gamma_{i+r})$, $i = 1,\ldots,r$, where the γ's are the generators of a $2r$–dimensional Clifford algebra, then c_i and π_i verify the relations (2.7).

The algebra (2.7) admits the (order reversing) involution $\bar{\cdot} : c_i \mapsto \bar{c}_i = \pi_i$, $\pi_i \mapsto \bar{\pi}_i = c_i$. The *anti*–BRST *operator* $\bar{s}_2$ is given by

$$\bar{\cdot} : s_2 \mapsto \bar{s}_2 \;, \qquad \bar{s}_2 = \frac{1}{2} C_{ij}{}^k c_k \pi^i \pi^j \quad \left(= \frac{1}{2} C_{ij}{}^k c_k \frac{\partial}{\partial c_i} \frac{\partial}{\partial c_j} \right) \;, \tag{2.8}$$

and it is also nilpotent. Denoting the space of the BRST q–cochains (1.5) by $C^q(\mathcal{G})$, it follows that

$$s_2 : C^q(\mathcal{G}) \to C^{q+1}(\mathcal{G}) \quad , \quad \bar{s}_2 : C^q(\mathcal{G}) \to C^{q-1}(\mathcal{G}) \quad . \tag{2.9}$$

The presence of a metric (δ_{ij}) on $\mathcal{G}$ allows us to introduce the $*$–operator $(* : C^q(\mathcal{G}) \to C^{r-q}(\mathcal{G}))$ in the standard way. On q–forms on G,

$$(*\psi) = \frac{1}{q!} \frac{1}{(r-q)!} \epsilon_{i_1 \ldots i_r} \psi^{i_1 \ldots i_q} \omega^{i_{q+1}} \wedge \ldots \wedge \omega^{i_r} \quad , \tag{2.10}$$

and

$$*^2 = (-1)^{q(r-q)} = (-1)^{q(r-1)} \, . \tag{2.11}$$

The scalar product of two LI q–forms on G, $\langle \cdot, \cdot \rangle : C^q(\mathcal{G}) \otimes C^q(\mathcal{G}) \to \mathbb{R}$ is then given by

$$\begin{aligned} \langle \psi', \psi \rangle :=& \int_G \psi' \wedge * \psi \\ =& \int_G \frac{1}{q!^2} \frac{1}{(r-q)!} \psi'_{j_1 \ldots j_q} \epsilon_{i_1 \ldots i_q j_{q+1} \ldots j_r} \psi^{i_1 \ldots i_q} \epsilon^{j_1 \ldots j_r} \omega^1 \wedge \ldots \wedge \omega^r \\ =& \int_G \frac{1}{q!^2} \epsilon^{j_1 \ldots j_q}_{i_1 \ldots i_q} \psi'_{j_1 \ldots j_q} \psi^{i_1 \ldots i_q} \omega^1 \wedge \ldots \wedge \omega^r = \frac{1}{q!} \psi'_{j_1 \ldots j_q} \psi^{j_1 \ldots j_q} \int_G \omega^1 \wedge \ldots \wedge \omega^r \end{aligned} \tag{2.12}$$

and, normalising the (compact) group volume $\int_G \omega^1 \wedge \ldots \wedge \omega^r$ to 1, reduces to (1.6). Clearly[3]

$$\langle \psi', \psi \rangle = \langle \psi, \psi' \rangle \quad , \quad \langle \psi, \psi \rangle > 0 \quad \forall \psi \neq 0 \quad . \tag{2.13}$$

The codifferential δ is introduced, as usual, as the adjoint of the exterior derivative d, *i.e.*, for a $(q-1)$–form ψ',

$$\begin{aligned} \langle d\psi', \psi \rangle =& \int_G d\psi' \wedge *\psi = (-1)^q \int_G \psi' \wedge d * \psi = (-1)^{q+(q-1)(r-q+1)} \int_G \psi' \wedge *(*d*\psi) \\ \equiv& \int_G \psi' \wedge *\delta\psi = \langle \psi', \delta\psi \rangle \quad , \end{aligned} \tag{2.14}$$

so that

$$\delta = (-1)^{r(q+1)+1} * d* \quad , \quad (d = (-1)^{r(q+1)} * \delta *) \quad , \quad \delta^2 = 0 \; . \tag{2.15}$$

[3]Using the c's to write ψ (eqn. (1.5)), rather than the ω's of (2.12), one might introduce a Berezin [34] integral measure to define $\langle \psi', \psi \rangle$ above as $\int dc^1 \ldots dc^r \psi'^{\dagger} \psi$ [14] for states ψ' and ψ of ghost numbers q and $r-q$ respectively. However, this leads to a product which is not positive definite [14] and, moreover, does not have the natural geometrical interpretation above.

The correspondence $\omega^i(g) \leftrightarrow c^i$, $d \leftrightarrow s_2$ above allows us to translate all this into the BRST language. First one checks, on any BRST q–cochain (1.5), that the basic operators c^i and π^i are transformed by $*$ according to

$$\pi^i = (-1)^{r(q+1)} * c^i * \quad , \quad c^i = (-1)^{r(q+1)+1} * \pi^i * \quad , \tag{2.16}$$

so that

$$\begin{aligned} *(c^{i_1} \dots c^{i_{2k}})* &= (-1)^{(r+1)q+k} \pi^{i_1} \dots \pi^{i_{2k}} \ , \ *(c^{i_1} \dots c^{i_{2k+1}})* = (-1)^{r(q+1)+k} \pi^{i_1} \dots \pi^{i_{2k+1}} \ , \\ *(\pi^{i_1} \dots \pi^{i_{2k}})* &= (-1)^{(r+1)q+k} c^{i_1} \dots c^{i_{2k}} \ , \ *(\pi^{i_1} \dots \pi^{i_{2k+1}})* = (-1)^{r(q+1)+k+1} c^{i_1} \dots c^{i_{2k+1}} \ . \end{aligned} \tag{2.17}$$

As a consequence of (2.16) one finds for $\psi' \in C^{q+1}(\mathcal{G})$, $\psi \in C^q(\mathcal{G})$,

$$\begin{aligned} \langle \psi', c^i \psi \rangle &= \int_G \psi' \wedge * c^i \psi = (-1)^{q(r-q)} \int_G \psi' \wedge * c^i * * \psi = (-1)^q \int_G \psi' \wedge \pi^i * \psi \\ &= \int_G \pi^i \psi' \wedge * \psi = \langle \pi^i \psi', \psi \rangle \quad , \end{aligned} \tag{2.18}$$

using the fact that π^i is a graded derivative and that $\psi' \wedge * \psi \equiv 0$. Thus, c^i and π^i are adjoints to each other with respect to the inner product $\langle \ , \ \rangle$ or, in other words, the involution $\bar{\cdot}$ defines the adjoint with respect to $\langle \ , \ \rangle$. Thus, $s_2 \sim d$ and (2.15) lead to

$$\overline{s}_2 = (-1)^{r(q+1)+1} * s_2 * \tag{2.19}$$

since

$$\begin{aligned} \delta &= (-1)^{r(q+1)+1} * d * \sim (-1)^{r(q+1)+1} * s_2 * = -(-1)^{r(q+1)+1} \frac{1}{2} C_{ij}{}^k * c^i c^j \pi_k * \\ &= -(-1)^{r(q+1)+1+(q-1)(r-q+1)+q(r-q)} \frac{1}{2} C_{ij}{}^k * c^i * * c^j * * \pi_k * = \frac{1}{2} C_{ij}{}^k \pi^i \pi^j c_k = \overline{s}_2 \ . \end{aligned} \tag{2.20}$$

The anticommutator of the nilpotent operators s_2 and $\overline{s}_2$ defines the Laplacian $\Delta \equiv W_2$, $W_2 : C^q(\mathcal{G}) \to C^q(\mathcal{G})$,

$$W_2 := \{s_2, \overline{s}_2\} = (s_2 + \overline{s}_2)^2 \quad . \tag{2.21}$$

The operators $W_2,\ s_2,\ \overline{s}_2$ generate the supersymmetry algebra Σ_2

$$[s_2, W_2] = 0 \quad , \quad [\overline{s}_2, W_2] = 0 \quad , \quad \{s_2, \overline{s}_2\} = W_2 \quad . \tag{2.22}$$

Σ_2 has the structure of a central extension of (s_2, $\overline{s}_2$) by W_2, the Laplacian being the central generator. The operator W_2 is invariant under the involution $\bar{\cdot}$ ($W_2 = \overline{W}_2$) and commutes with $*$, since

$$\begin{aligned} * W_2 * &= *(s_2 \overline{s}_2 + \overline{s}_2 s_2)* = (-1)^{(q-1)(r-q+1)} * (s_2 * * \overline{s}_2 + \overline{s}_2 * * s_2)* \\ &= (-1)^{(q-1)(r-q+1)+r(q-1)+1+rq} (\overline{s}_2 s_2 + s_2 \overline{s}_2) = (-1)^{q(r-q)} W_2 \quad , \end{aligned} \tag{2.23}$$

which, with the help of (2.11), implies $[W_2, *] = 0$. Then, as in the standard Hodge theory on compact Riemannian manifolds, we have

Lemma 2.1
A BRST cochain ψ is W_2–harmonic, $W_2\psi = 0$, iff it is s_2 *and* $\overline{s}_2$–closed.

Proof. It is clear that if $s_2\psi = 0 = \overline{s}_2\psi$, then $W_2\psi = 0$. Now, if $W_2\psi = 0$,

$$0 = \langle \psi, W_2\psi \rangle = \langle \psi, (s_2\overline{s}_2 + \overline{s}_2 s_2)\psi \rangle = \langle \overline{s}_2\psi, \overline{s}_2\psi \rangle + \langle s_2\psi, s_2\psi \rangle \quad ; \tag{2.24}$$

from (2.13) easily follows that both terms have to be zero and hence $s_2\psi = 0 = \overline{s}_2\psi$.

Theorem 2.1
Each BRST cochain ψ admits the Hodge decomposition

$$\psi = s_2\alpha + \overline{s}_2\beta + \gamma \quad , \tag{2.25}$$

where γ is W_2–harmonic (the proof of theorem. 3.1 below includes this case).

To find the algebraic meaning of W_2, let us write the generators X_i on ghost space as

$$X_i \equiv -C_{ij}{}^k c^j \pi_k \quad . \tag{2.26}$$

They act on BRST cochains in the same way as the Lie derivatives with respect to the LI vector fields on G act on LI forms on G:

$$X_i c^k = -C_{ij}{}^k c^j \ , \tag{2.27}$$

(*cf.* $L_{X_i}\omega^k = -C_{ij}{}^k\omega^j$, in which $X_i \in \mathfrak{X}^L(G)$ and $\omega \in \mathfrak{X}^{*L}(G)$). The X_i in (2.26) are in the adjoint representation of $\mathcal{G}$ and satisfy $\overline{X}_i = -X_i$ and $*X_i = X_i*$. *Invariant states* are those for which $X_i\psi = 0$, $i = 1, \ldots, r$.

In terms of X_i, the operators s_2 and $\overline{s}_2$ may be written as

$$s_2 = \frac{1}{2}c^i X_i \ , \qquad \overline{s}_2 = -\frac{1}{2}\pi^j X_j \ . \tag{2.28}$$

Using the fact that c^i and π^j transform in the adjoint representation,

$$X_k c^i = -C_{kr}{}^i c^r \quad , \quad X_k \pi^i = -C_{kr}{}^i \pi^r \ , \tag{2.29}$$

it is easy to see that[4]

$$W_2 = -\frac{1}{2}\mathcal{C}^{(2)} = -\frac{1}{2}\delta^{ij}X_i X_j \quad , \tag{2.30}$$

i.e. the Laplace–type operator is proportional to the second order Casimir operator of the algebra.

[4] Taking advantage of the Cartan formalism by means of the equivalences $s_2 \sim d$, $\overline{s}_2 \sim (-1/2)L_j i_j$ (where i_j indicates the inner product) and $X_j \sim L_j$, we may rapidly find $W_2 \sim (-1/2)[dL_j i_j + L_j i_j d] = (-1/2)L_j[di_j + i_j d] = (-1/2)L_j L_j$.

Remark. The expression for W_2 in [12, 13, 14] contains more terms due to the fact that these authors consider $\rho \neq 0$ in general. But, as noticed in [14], $\rho = 0$ is the only possibility if we restrict ourselves to *non–trivial* harmonic states. In fact, we prove here that this is a direct consequence of Whitehead's lemma (2.4). Let τ be the operator defined by its action on (V–valued) q–cochains ψ through

$$(\tau\psi)^A_{i_1\ldots i_{q-1}} = k^{ij}\rho(X_i)^A_B\psi^B_{ji_1\ldots i_{q-1}} \quad . \tag{2.31}$$

It may be verified that

$$[(s_\rho\tau + \tau s_\rho)\psi]^A_{i_1\ldots i_q} = \psi^B_{i_1\ldots i_q}\mathcal{C}^{(2)}(\rho)^A_B \quad , \tag{2.32}$$

where $\mathcal{C}^{(2)}(\rho)^A_B \equiv k^{ij}\rho(X_i)^A_C\rho(X_j)^C_B$ is the Casimir operator for the representation ρ, and hence proportional to δ^A_B. It then follows that for any $\rho \neq 0$ q–cocycle ψ ($s_\rho\psi = 0$)

$$s_\rho(\tau\psi\mathcal{C}^{(2)}(\rho)^{-1}) \propto \psi \tag{2.33}$$

i.e., ψ is a (trivially harmonic state) coboundary generated by a $(q-1)$–cochain proportional to $\tau\psi\mathcal{C}^{(2)}(\rho)^{-1}$, *q.e.d.* Hence, any non–trivial BRST–invariant state ψ ($s_2\psi = 0$, $\psi \neq s_2\varphi$) is a $\mathcal{G}$ singlet and, as a consequence of Th. 2.1, its class contains a unique W_2 harmonic representative.

From (2.30) we also deduce the following

Lemma 2.2
A state ψ is W_2–harmonic iff it is invariant[5], $X_i\psi = 0$.

Proof. If ψ is invariant, $W_2\psi = -\frac{1}{2}\delta^{ij}X_iX_j\psi = 0$. If ψ is W_2–harmonic,

$$0 = \langle\psi, W_2\psi\rangle = -\frac{1}{2}\langle\psi, \delta^{ij}X_iX_j\psi\rangle = \frac{1}{2}\delta^{ij}\langle X_i\psi, X_j\psi\rangle \tag{2.34}$$

and $X_j\psi = 0$, since $\langle\ ,\ \rangle$ is non–degenerate, *q.e.d.* In fact, if ψ is invariant, ψ is both s_2 and $\bar{s}_2$ closed by (2.28).

Corollary 2.1
Each non–trivial element in the cohomology ring $H^*(\mathcal{G})$ may be represented by an invariant state.

Proof. Let $\psi \in Z(\mathcal{G})$ be nontrivial. Hence its decomposition has the form

$$\psi = s_2\alpha + \gamma \quad . \tag{2.35}$$

Therefore $\psi - s_2\alpha$ is in the cohomology class of ψ and is harmonic (and hence invariant).

[5]The CE analogues to the BRST q–cochains, the LI q–forms on G, automatically satisfy $L_{X^R}\psi = 0$, since the RI vector fields X^R on G generate the *left* transformations. The invariance under the right transformations ($L_X\psi = 0$ where X is a LI vector field, or $X\psi = 0$ in the BRST formulation) is an *additional* condition. Thus, invariance above really means *bi-invariance* (under the left and right group translations) in the CE formulation of Lie algebra cohomology.

3 Higher order BRST and anti–BRST operators

3.1 Invariant tensors

The considerations of the previous section rely on the nilpotent operator s_2 and its adjoint, both constructed out of the structure constants C_{ijk}. The latter determine a skew-symmetric tensor of order three which can be seen as a third order cocycle $C = C_{ijk}c^i c^j c^k$ and, additionally, is invariant under the action of the Lie algebra generators X_k. Indeed, acting on C with the X's one gets a sum of three terms, in each of which one of the indices of C is transformed in the adjoint representation and the statement of invariance is equivalent to the Jacobi identity. Notice that we need not saturate every index of C_{ijk} with the same type of variable in order to get an invariant quantity —it suffices that each type of variable transforms in the adjoint representation (for example, s_2 in (1.1), which is also invariant, involves saturating two c's and one π).

The cohomology of simple Lie algebras contains, besides the three cocycle C above, other, higher order skewsymmetric tensors with similar properties. As mentioned in the introduction, any compact simple Lie algebra $\mathcal{G}$ of rank l has l primitive cocycles given by skew-symmetric tensors $\Omega^{(2m_s-1)}_{i_1 i_2 \ldots i_{2m_s-1}}$ $(s = 1, \ldots, l)$, associated to the l Casimir–Racah primitive invariants of rank m_s [20, 21, 22, 24]. Their invariance is expressed by the equation

$$\sum_{j=1}^{2m_s-1} C_{bi_j}{}^a \Omega^{(2m_s-1)}_{i_1 i_2 \ldots \hat{i}_j a i_{j+1} \ldots i_{2m_s-1}} = 0 \quad . \tag{3.1}$$

Due to the MC equations, the above relation implies that $\Omega^{(2m_s-1)}_{i_1 \ldots i_{2m_s-1}} \omega^{i_1} \ldots \omega^{i_{2m_s-1}}$ is a cocycle for the Lie algebra coboundary operator (2.3) for ρ=0 (in the language of forms, this is equivalent to saying that any bi–invariant form is closed, and hence a CE cocycle). The existence of these cocycles is related to the topology of the corresponding group manifold, in particular to the odd-sphere product structure that the simple compact group manifolds have from the point of view of real homology (see *e.g.* [32, 35, 36, 37, 33]). We may use the correspondence $c^i \leftrightarrow \omega^i$ and the discussion after theorem 2.1 to move freely from the CE approach to the BRST one here.

Let us consider for definiteness the case of $su(n)$, for which $m_1 = 2$, $m_2 = 3, \ldots, m_l = n$ and there exist l=$(n-1)$ different primitive skew–symmetric tensors of rank $3, 5, \ldots, 2n-1$. Consider, for a given m, the $(2m-1)$–form

$$\Omega^{(2m-1)} = \frac{1}{(2m-1)!} \mathrm{Tr}(\theta \wedge \overset{2m-1}{\cdots} \wedge \theta) \,, \tag{3.2}$$

where $\theta \equiv \omega^i T_i$ and $T_i \in \mathcal{G}$ is in the defining representation of $su(n)$. Since $d\Omega^{(2m-1)} = 0$, the coordinates of $\Omega^{(2m-1)}$ provide a $(2m-1)$–cocycle on $su(n)$. One can show (see *e.g.* [23]) that

$$\Omega^{(2m-1)}_{\rho i_2 \ldots i_{2m-2} \sigma} = \frac{1}{(2m-3)!} k_{\rho l_1 \ldots l_{m-1}} C_{j_2 j_3}{}^{l_1} \ldots C_{j_{2m-2}\sigma}{}^{l_{m-1}} \epsilon^{j_2 \ldots j_{2m-2}}_{i_2 \ldots i_{2m-2}} \,, \tag{3.3}$$

is a skew–symmetric tensor, where

$$k_{\rho l_1 \ldots l_{m-1}} = \mathrm{sTr}(T_\rho T_{l_1} \ldots T_{l_{m-1}}) \tag{3.4}$$

is a symmetric invariant tensor given by the symmetrised trace of a product of m generators (its invariance can be expressed by an equation similar to (3.1)). Symmetric invariant tensors $k_{i_1 \ldots i_m}$ give rise to Casimir-Racah operators

$$\mathcal{C}^{(m)} = k^{i_1 \ldots i_m} X_{i_1} \ldots X_{i_m} \tag{3.5}$$

which commute with the generators; $\mathcal{C}^{(2)}$ is the standard quadratic Casimir operator.

3.2 Higher order operators

The above family of cocycles Ω can be used to construct *higher–order* BRST *operators* [24, 31]. To each invariant tensor of rank m_s corresponds a BRST operator s_{2m_s-2} which, in terms of the coordinates $\Omega_{i_1 \ldots i_{2m_s-2}}{}^{\sigma}$ of the $(2m_s-1)$–cocycle (3.3), is given by

$$s_{2m_s-2} = -\frac{1}{(2m_s-2)!} \Omega^{(2m_s-1)}_{i_1 \ldots i_{2m_s-2}}{}^{\sigma} c^{i_1} c^{i_2} \ldots c^{i_{2m_s-2}} \pi_\sigma \ . \tag{3.6}$$

These operators are particularly interesting in view of the property

$$\{s_{2m_s-2},\ s_{2m_{s'}-2}\} = 0 \quad , \quad s, s' = 1, \ldots, l \quad ; \tag{3.7}$$

i.e., they are nilpotent and anticommute (see [24] for a proof).

For each m_s, $s > 1$, we may look at s_{2m-2} [6] as a *higher order coboundary* operator, $s_{2m-2} : C^q(\mathcal{G}) \to C^{q+(2m-3)}(\mathcal{G})$. The analogue of the MC equation (2.6) for s_{2m-2} is given by

$$s_{2m-2} c^a = -\frac{1}{(2m-2)!} \Omega^{(2m-1)}_{i_1 \ldots i_{2m-2}}{}^{a} c^{i_1} c^{i_2} \ldots c^{i_{2m-2}} \tag{3.8}$$

which may also be written as

$$s_{2m-2} c = -\frac{1}{(2m-2)!} [c, \overset{2m-2}{\cdots}, c] \quad , \tag{3.9}$$

where $[c, \overset{2m-2}{\cdots}, c] := c^{i_1} \ldots c^{i_{2m_2}} [T_{i_1}, \ldots, T_{i_{2m-2}}]$ and the higher–order structure constants of the $(2m-2)$–bracket [24] are given by the $(2m-1)$ cocycle, *i.e.*

$$[T_{i_1}, \ldots, T_{i_{2m-2}}] = \Omega^{(2m-1)}_{i_1 \ldots i_{2m-2}}{}^{a} T_a \quad . \tag{3.10}$$

Using (3.9), the nilpotency of s_{2m-2} follows from the higher order Jacobi identity

$$s^2_{2m_2} c = -\frac{1}{(2m-2)!} \frac{1}{(2m-3)!} [c, \overset{2m-3}{\cdots}, c, [c, \overset{2m-2}{\cdots}, c]] = 0 \tag{3.11}$$

[6] We shall often write m for m_s henceforth.

which the r.h.s. of (3.11) satisfies as a consequence of $\Omega^{(2m-1)\ \ a}_{i_1\ldots i_{2m-2}}$ being a cocycle.

Moreover, for each $\mathcal{G}$ we may introduce the *complete BRST operator* $\mathbf{s}$ [24]

$$\begin{aligned}\mathbf{s} =&-\frac{1}{2}C_{j_1j_2}{}^{\sigma}c^{j_1}c^{j_2}\pi_\sigma - \ldots - \frac{1}{(2m_s-2)!}\Omega^{(2m_s-1)\ \ \sigma}_{j_1\ldots j_{2m_s-2}}c^{j_1}\ldots c^{j_{2m_s-2}}\pi_\sigma - \ldots \\ &-\frac{1}{(2m_l-2)!}\Omega^{(2m_l-1)\ \ \sigma}_{j_1\ldots j_{2m_l-2}}c^{j_1}\ldots c^{j_{2m_l-2}}\pi_\sigma \equiv s_2 + \ldots + s_{2m_s-2} + \ldots + s_{2m_l-2}\end{aligned} \tag{3.12}$$

This operator is nilpotent, and its terms have (except for some additional ones that break the generalised Jacobi identities which are at the core of the nilpotency of s_{2m_s-2}) the same structure as those which appear in closed string theory [26] and lead to a strongly homotopy algebra [25]. In fact, the higher order structure constants (which here have definite values and a geometrical meaning as higher order cocycles of $\mathcal{G}$) correspond to the string correlation functions giving the string couplings. Since the expression for $\mathbf{s}$ in the homotopy Lie algebra that underlies closed string theory already includes a term of the form $f^{\sigma}_{j_1}c^{j_1}\pi_\sigma$, f nilpotent, $\mathbf{s}^2=0$ is not satisfied (as it is for (3.12)) by means of a sum of independently satisfied Jacobi identities, and in particular the $C_{ij}{}^k$ do not satisfy the Jacobi identity and hence do not define a Lie algebra.

For each s_{2m-2} we now introduce its adjoint *anti*–BRST *operator* $\overline{s}_{2m-2}$,

$$\begin{aligned}\overline{s}_{2m-2} &= -\frac{1}{(2m-2)!}\Omega^{(2m-1)\ \ \sigma}_{i_1\ldots i_{2m-2}}c_\sigma\pi^{i_{2m-2}}\ldots\pi^{i_1} \\ &= -\frac{(-1)^{m-1}}{(2m-2)!}\Omega^{(2m-1)\ \ \sigma}_{i_1\ldots i_{2m-2}}c_\sigma\pi^{i_1}\ldots\pi^{i_{2m-2}}\,.\end{aligned} \tag{3.13}$$

Each pair $(s_{2m-2},\,\overline{s}_{2m-2})$ allows us to construct a *higher-order Laplacian* W_{2m-2}

$$W_{2m-2} = (s_{2m-2}+\overline{s}_{2m-2})^2 = s_{2m-2}\overline{s}_{2m-2} + \overline{s}_{2m-2}s_{2m-2} \quad . \tag{3.14}$$

Clearly, s_{2m-2}, $\overline{s}_{2m-2}$ and W_{2m-2} all commute with the generators X_i, and we have the following

Lemma 3.1

For each $s=1,\ldots,l$, the higher order BRST and anti-BRST operators s_{2m_s-2} and $\overline{s}_{2m_s-2}$, together with their associated Laplacian W_{2m_s-2} define the superalgebra Σ_{m_s}

$$[s_{2m_s-2},W_{2m_s-2}]=0 \quad , \quad [\overline{s}_{2m_s-2},W_{2m_s-2}]=0 \quad , \quad \{s_{2m_s-2},\overline{s}_{2m_s-2}\}=W_{2m_s-2} \quad , \tag{3.15}$$

which has a central extension structure.

For $s=1$, $m_1=2$, $W_2=\Delta$ and the above expressions reproduce (1.1), (2.6), (2.8) and (2.22).

The BRST (anti-BRST) operator s_{2m-2} ($\overline{s}_{2m-2}$), acting on a monomial in the c's, raises (lowers) its ghost number by $2m-3$ while W_{2m-2} leaves the ghost number invariant and is self-adjoint. We notice that all terms in s_{2m-2} ($\overline{s}_{2m-2}$) contain one $(2m-2)$ π, and that the term with the maximum number of π's in W_{2m-2} contains (at most) $2m-2$ of them. This is so because the two terms with $2m-1$ π's (from $s_{2m-2}\overline{s}_{2m-2}$, $\overline{s}_{2m-2}s_{2m-2}$) cancel, as one can verify. The BRST operator s_{2m-2} annihilates all states of ghost number $q > r-2m+2$ (since the product of more than r c's necessarily vanishes) as well as zeroth order states. Similarly, $\overline{s}_{2m-2}$ annihilates states of ghost number $q < 2m-2$ and the top state $c_1 \ldots c_r$. It follows that zero and r–ghost number states are both W_{2m-2}–harmonic.

Let us establish now the relation between the $\bar{\cdot}$ operation (the adjoint with respect to the inner product in (2.12)) and the conjugation by the Hodge $*$ operator, as these apply to s_{2m-2}.

Lemma 3.2
The following equalities hold on any state (BRST-cochain) of ghost number q,

$$\overline{s}_{2m-2} = (-1)^{r(q+1)+1} * s_{2m-2} * \, , \; s_{2m-2} = (-1)^{r(q+1)} * \overline{s}_{2m-2} * \, , \; s_{2m-2} * = (-1)^q * \overline{s}_{2m-2} \; ; \tag{3.16}$$

notice that the sign factors do not depend on m and hence they coincide with those of (2.15). The proof is straightforward, using (2.11), (2.17), where care should be taken to substitute the ghost numbers actually 'seen' by the operators.

Although we shall not use them here we also introduce, for the sake of completeness, the V–valued higher order coboundary operators $s_{\rho(2m_s-2)}$ for a non-trivial representation $\rho \in \mathrm{End}\, V$ of the $(2m_s-2)$-algebra. They are given by

$$s_{\rho(2m_s-2)} = c^{i_1} \ldots c^{i_{2m_s-3}} \rho(X_{i_1}) \ldots \rho(X_{i_{2m_s-3}}) - \frac{1}{(2m_s-2)!} \Omega^{(2m_s-1)}_{i_1 \ldots i_{2m_s-2}}{}^{\sigma} c^{i_1} c^{i_2} \ldots c^{i_{2m_s-2}} \pi_\sigma \quad . \tag{3.17}$$

It may be seen that the nilpotency of $s_{\rho(2m_s-2)}$ is guaranteed by the fact that the skewsymmetric product of $(2m_s$–$2)$ ρ's, which defines the multibracket $(2m_s-2)$-algebra for an appropriate ρ (eqn. (3.10) with $T \to \rho$), satisfies the corresponding generalised Jacobi identity as before.

3.3 Higher order Hodge decomposition and representations of Σ_{m_s}

Let us now look at the irreducible representations of (3.15), which have the same structure as the supersymmetry algebra. Since W_{2m-2} commutes with s_{2m-2}, $\overline{s}_{2m-2}$, each multiplet of states will have a fixed W_{2m-2}–eigenvalue. Let us call γ a W_{2m-2}–*harmonic state* iff $W_{2m-2}\gamma = 0$. Then lemma 2.1 transcribes trivially to the present higher order case so that γ is harmonic iff it is s_{2m-2} and $\overline{s}_{2m-2}$–closed. Hence a harmonic state γ is a singlet of Σ_m. We may also extend theorem 2.1 and prove the following

Theorem 3.1 (*Higher order Hodge decomposition*)
Each BRST cochain ψ admits a unique decomposition

$$\psi = s_{2m-2}\alpha + \overline{s}_{2m-2}\beta + \gamma \tag{3.18}$$

where γ is W_{2m-2}–harmonic.

Proof. We denote by $\mathcal{S}$ the space of all states (*i.e.* skewsymmetric polynomials in the c's), $\mathcal{K}_{2m-2}$ the kernel of W_{2m-2} (W_{2m-2}–harmonic space) and $\mathcal{K}^{\perp}_{2m-2}$ the complement of $\mathcal{K}_{2m-2}$ in $\mathcal{S}$. Let $P^{(0)}_W$ be the projector from $\mathcal{S}$ to $\mathcal{K}_{2m-2}$. Let $\psi \in \mathcal{S}$; then, $(1 - P^{(0)}_W)\psi$ lies in $\mathcal{K}^{\perp}_{2m-2}$. However, since the restriction of W_{2m-2} to $\mathcal{K}^{\perp}_{2m-2}$ is invertible, there exists a unique ϕ in $\mathcal{K}^{\perp}_{2m-2}$ such that $(1 - P^{(0)}_W)\psi = W_{2m-2}\phi$, from which we get

$$\begin{aligned} \psi &= W_{2m-2}\phi + P^{(0)}_W\psi \\ &= s_{2m-2}(\overline{s}_{2m-2}\phi) + \overline{s}_{2m-2}(s_{2m-2}\phi) + P^{(0)}_W\psi \quad , \end{aligned} \tag{3.19}$$

which provides the desired decomposition of ψ with $\alpha = \overline{s}_{2m-2}\phi$, $\beta = s_{2m-2}\phi$ and $\gamma = P^{(0)}_W\psi$, *q.e.d.*

To complete the analysis of the irreducible representations of Σ consider now an eigenstate χ of W_{2m-2} for non-zero (and hence positive) eigenvalue w, $W_{2m-2}\chi = w\chi$, $w > 0$. This gives rise to the states

$$\phi \equiv s_{2m-2}\chi \quad , \qquad \psi \equiv \overline{s}_{2m-2}\chi \quad , \qquad \sigma \equiv s_{2m-2}\overline{s}_{2m-2}\chi \quad . \tag{3.20}$$

Further applications of s_{2m-2} or $\overline{s}_{2m-2}$ produce linear combinations of the above states, for example $\overline{s}_{2m-2}s_{2m-2}\chi = W_{2m-2}\chi - s_{2m-2}\overline{s}_{2m-2}\chi = w\chi - \sigma$ *etc.* The quartet $\{\chi, \phi, \psi, \sigma\}$ collapses to a doublet if either $s_{2m-2}\chi = 0$ or $\overline{s}_{2m-2}\chi = 0$. In this case, χ is the Clifford vacuum and s_{2m-2}, or $\overline{s}_{2m-2}$, respectively, plays the role of the annihilation operator. Let χ be neither s_{2m-2} nor $\overline{s}_{2m-2}$-closed. The state σ of (3.20) is, by construction, s_{2m-2}-closed. Then, we can always choose a linear combination of χ and σ that is $\overline{s}_{2m-2}$-closed. Indeed, for the $\{\chi, \phi, \psi, \sigma\}$ of (3.20) we easily compute

$$\begin{aligned} \|\phi\|^2 + \|\psi\|^2 &= \langle s_{2m-2}\chi\, , \, s_{2m-2}\chi\rangle + \langle \overline{s}_{2m-2}\chi\, , \, \overline{s}_{2m-2}\chi\rangle \\ &= \langle \chi\, , \, W_{2m-2}\chi\rangle \quad = \quad w \; , \end{aligned} \tag{3.21}$$

where we have taken $\|\chi\|^2 \equiv \langle \chi\, , \, \chi\rangle = 1$. Setting

$$q = \sqrt{w}\, , \qquad \|\phi\| = q\sin\theta\, , \qquad \|\psi\| = q\cos\theta\, , \tag{3.22}$$

we find that the following linear combinations

$$\chi' = \frac{1}{q\sin\theta}(q\chi - q^{-1}\sigma)\, , \qquad \sigma' = \frac{1}{q^2\cos\theta}\sigma\, , \qquad \phi' = \frac{1}{q\sin\theta}\phi\, , \qquad \psi' = \frac{1}{q\cos\theta}\psi\, , \tag{3.23}$$

form an orthonormal set, with the doublet $\{\chi'\, , \, \phi'\}$ satisfying

$$\overline{s}_{2m-2}\chi' = 0 \quad , \quad s_{2m-2}\chi' = q\phi' \quad ; \quad s_{2m-2}\phi' = 0 \quad , \quad \overline{s}_{2m-2}\phi' = q\chi' \quad , \tag{3.24}$$

and similarly for $\{\psi', \sigma'\}$, *i.e.* the two doublets decouple. Notice that θ in (3.22), $0 < \theta < \pi/2$, is the angle between χ and σ in the χ-σ plane:

$$\langle \chi, \sigma \rangle = q^2 \cos^2 \theta = \|\chi\| \cdot \|\sigma\| \cos\theta \ . \tag{3.25}$$

Once in the primed basis of (3.23), $s_{2m-2}\overline{s}_{2m-2}\xi$ (where ξ stands for any of the four primed states above) is equal to either $w\xi$ or 0 and hence, $s_{2m-2}\overline{s}_{2m-2}$ commutes with all operators that commute with W_{2m-2} (similarly for $\overline{s}_{2m-2}s_{2m-2}$). Thus, the representation of the different superalgebras Σ_m fall, in all cases, into singlets (harmonic states) and pairs of doublets. Singlets and doublets here are the trivial analogues of the 'short' (massless) and 'long' (massive) multiplets of the standard supersymmetry algebra.

Owing to the particular importance of harmonicity, we investigate the relation between the kernel of a higher-order Laplacian and that of $W_2 \propto \mathcal{C}^{(2)}$. To this end, we rewrite s_{2m-2} as (Greek indices below also range in $1, \ldots, r \equiv \dim \mathcal{G}$).

$$\begin{aligned}
s_{2m-2} &= -\frac{1}{(2m-2)!}\Omega^{(2m-1)}_{i_1\ldots i_{2m-2}}{}^{\sigma} c^{i_1}\ldots c^{i_{2m-2}}\pi_\sigma \\
&= -\frac{1}{(2m-2)!}k_{j_1\ldots j_{m-1}}{}^{\sigma} C_{\rho i_2}{}^{j_1}\ldots C_{i_{2m-3}i_{2m-2}}{}^{j_{m-1}} c^\rho c^{i_2}\ldots c^{i_{2m-2}}\pi_\sigma \\
&= \frac{1}{(2m-2)!}\Big(\sum_{r=2}^{m-1} k_{i_2 j_2 \ldots \widehat{j_r}\alpha\ldots j_{m-1}}{}^{\sigma} C_{\rho j_r}{}^{\alpha} C_{i_3 i_4}{}^{j_2}\ldots C_{i_{2r-1}i_{2r}}{}^{j_r}\ldots C_{i_{2m-3}i_{2m-2}}{}^{j_{m-1}} \\
&\quad + k_{i_2 j_2\ldots j_{m-1}}{}^{\alpha} C_{\alpha\rho}{}^{\sigma} C_{i_3 i_4}{}^{j_2}\ldots C_{i_{2m-3}i_{2m-2}}{}^{j_{m-1}}\Big) c^\rho c^{i_2}\ldots c^{i_{2m-2}}\pi_\sigma \\
&= \frac{1}{(2m-2)!}k_{\beta j_1\ldots j_{m-2}}{}^{\alpha} C_{i_1 i_2}{}^{j_1}\ldots C_{i_{2m-5}i_{2m-4}}{}^{j_{m-2}} c^\beta c^{i_1}\ldots c^{i_{2m-4}} X_\alpha \ ,
\end{aligned} \tag{3.26}$$

where the invariance of $k_{j_1\ldots j_{m-1}}{}^{\sigma}$ has been used in the third line and the Jacobi identity in the last equality. Similarly, $\overline{s}_{2m-2}$ may be written as

$$\overline{s}_{2m-2} = -(-1)^m \frac{1}{(2m-2)!}k_{\beta j_1\ldots j_{m-2}}{}^{\alpha} C_{i_1 i_2}{}^{j_1}\ldots C_{i_{2m-5}i_{2m-4}}{}^{j_{m-2}} \pi^\beta \pi^{i_1}\ldots\pi^{i_{2m-4}} X_\alpha \ , \tag{3.27}$$

This proves the following

Lemma 3.3
The higher order BRST and anti–BRST operators s_{2m-2}, $\overline{s}_{2m-2}$ may be written as

$$s_{2m-2} = \Omega^\alpha X_\alpha \quad , \quad \overline{s}_{2m-2} = -\overline{\Omega}^\alpha X_\alpha \quad , \tag{3.28}$$

where

$$\Omega^\alpha \equiv \frac{1}{(2m-2)!}k_{\beta j_1\ldots j_{m-2}}{}^{\alpha} C_{i_1 i_2}{}^{j_1}\ldots C_{i_{2m-5}i_{2m-4}}{}^{j_{m-2}} c^\beta c^{i_1}\ldots c^{i_{2m-4}} \quad , \tag{3.29}$$

$$\overline{\Omega}^\alpha \equiv (-1)^m \frac{1}{(2m-2)!}k_{\beta j_1\ldots j_{m-2}}{}^{\alpha} C_{i_1 i_2}{}^{j_1}\ldots C_{i_{2m-5}i_{2m-4}}{}^{j_{m-2}} \pi^\beta \pi^{i_1}\ldots\pi^{i_{2m-4}} \quad . \tag{3.30}$$

For $m=2$ one gets $\Omega^\alpha = \frac{1}{2}c^\alpha$, $\overline{\Omega}^\alpha = \frac{1}{2}\pi^\alpha$ and the expression (2.28) for s_2, $\overline{s}_2$ is recovered. As a W_2–harmonic state is invariant, the above relation shows that the kernel of W_2 is contained in the kernel of W_{2m_s-2} for all m_s, $s=2,\ldots,l$. It follows from (3.28) that an invariant state is s_{2m-2} and $\overline{s}_{2m-2}$ closed, and hence the following lemma

Lemma 3.4
Every invariant state is W_{2m-2} harmonic.

In the particular realization of the Σ_{m_s} algebra (3.15) in terms of ghosts and antighosts given in (3.6), (3.13), W_{2m_s-2} is ghost number preserving and commutes with the Lie algebra generators X_i. There exists therefore a basis of the c's in which W_{2m_s-2}, $s=1,\ldots,l$ is diagonal. For a fixed ghost number q, the $\begin{pmatrix} r \\ q \end{pmatrix}$ independent monomials $c_{i_1}\ldots c_{i_q}$ transform as the fully antisymmetric part of the q–th tensor power of the adjoint representation of $\mathcal{G}$. This antisymmetric part is $\mathcal{G}$-reducible and W_{2m_s-2} will have a fixed eigenvalue in each of its irreducible components (which will change in general when going from one irreducible representation to another of the same or different ghost number). s_{2m_s-2} and $\overline{s}_{2m_s-2}$ connect states belonging to the same $\mathcal{G}$-irreducible representation and with the same W_{2m_s-2}–eigenvalue (but of different ghost number) and such states will fall into one of the Σ_{m_s} multiplets (singlets or doublets) discussed above. As the generators X_i commute with $*$, the $\mathcal{G}$-irreducible representation decomposition pattern will be symmetrical under $q \to r-q$.

The l Casimir–Racah operators $\mathcal{C}^{(m_s)}$, take fixed eigenvalues within each $\mathcal{G}$-irreducible component, which is uniquely labelled by them. The same irreducible representation may appear more than once, with equal or with different ghost numbers; the Casimirs will not distinguish among these different copies of the same irreducible representation. As mentioned, $W_2 \equiv \Delta = -\frac{1}{2}\mathcal{C}^{(2)}$ (eqn. (2.30)). An important question that naturally arises is whether W_{2m_s-2} also reduces, for $s>2$, to some higher order Casimir-Racah operator or, more generally, to a sum of products of them. Since W_{2m_s-2} commutes with the X's (realized in ghost space via (2.26)) an equivalent question is whether it belongs to the universal enveloping algebra $\mathcal{U}(\mathcal{G})$ of $\mathcal{G}$. The answer is negative and we address this point in the next section, working out in full the $su(3)$ case.

4 The case of $su(3)$

We opt here for mild departures from our previous conventions: the generators will now be chosen hermitian so as to work with real eigenvalues and the normalisation of all operators is such that fractional eigenvalues are avoided.

4.1 Invariant tensors and operators

The $su(3)$ algebra $[T_i, T_j] = i f_{ij}{}^k T_k\,, \quad i=1,\ldots,8$, is determined by the non-zero structure constants $f_{ij}{}^k$ which are reproduced for convenience:

Table 4.1 Non-zero structure constants for $su(3)$

$f_{123} = 1$	$f_{147} = 1/2$	$f_{156} = -1/2$
$f_{246} = 1/2$	$f_{257} = 1/2$	$f_{345} = 1/2$
$f_{367} = -1/2$	$f_{458} = \sqrt{3}/2$	$f_{678} = \sqrt{3}/2$

These are also the coordinates of the $su(3)$ three-cocycle. The well known third order symmetric tensor d_{ijk} (table 4.2)

Table 4.2 Non-zero components of the symmetric invariant d_{ijk} for $su(3)$

$d_{118} = 1/\sqrt{3}$	$d_{228} = 1/\sqrt{3}$	$d_{338} = 1/\sqrt{3}$	$d_{888} = -1/\sqrt{3}$
$d_{448} = -1/(2\sqrt{3})$	$d_{558} = -1/(2\sqrt{3})$	$d_{668} = -1/(2\sqrt{3})$	$d_{778} = -1/(2\sqrt{3})$
$d_{146} = 1/2$	$d_{157} = 1/2$	$d_{247} = -1/2$	$d_{256} = 1/2$
$d_{344} = 1/2$	$d_{355} = 1/2$	$d_{366} = -1/2$	$d_{377} = -1/2$

gives the third–order Casimir–Racah operator. From d_{ijk} and (3.3) one finds the $su(3)$ five-cocycle coordinates [23] (table 4.3).

Table 4.3 Non-zero coordinates of the $su(3)$ five-cocycle

$\Omega_{12345} = 1/4$	$\Omega_{12367} = 1/4$	$\Omega_{12458} = \sqrt{3}/12$
$\Omega_{12678} = -\sqrt{3}/12$	$\Omega_{13468} = -\sqrt{3}/12$	$\Omega_{13578} = -\sqrt{3}/12$
$\Omega_{23478} = \sqrt{3}/12$	$\Omega_{23568} = -\sqrt{3}/12$	$\Omega_{45678} = -\sqrt{3}/6$

The Casimirs C_2 and C_3,

$$C_2 = T^i T_i \quad , \quad C_3 = d^{ijk} T_i T_j T_k \quad , \tag{4.1}$$

are related to the operators (3.5) simply by

$$C_2 = -\mathcal{C}^{(2)} \quad , \quad C_3 = -i\mathcal{C}^{(3)} \quad . \tag{4.2}$$

The antisymmetric cocycles, on the other hand, give rise to the BRST and anti-BRST operators s_2, $\overline{s}_2$, s_4, $\overline{s}_4$ (see also (3.13))

$$s_2 = -\frac{1}{2} f_{ij}{}^k c^i c^j \pi_k \quad , \quad s_4 = -\frac{1}{4!} \Omega_{i_1 i_2 i_3 i_4}{}^{\sigma} c^{i_1} c^{i_2} c^{i_3} c^{i_4} \pi_\sigma \quad ; \tag{4.3}$$

$$s_2{}^2 = s_4{}^2 = 0 = \overline{s}_2{}^2 = \overline{s}_4{}^2 \quad , \quad s_2 s_4 + s_4 s_2 = 0 = \overline{s}_2 \overline{s}_4 + \overline{s}_4 \overline{s}_2 \ . \tag{4.4}$$

The corresponding Laplacians are

$$W_2 = (s_2 + \overline{s}_2)^2 = s_2 \overline{s}_2 + \overline{s}_2 s_2 \quad , \quad W_4 = (s_4 + \overline{s}_4)^2 = s_4 \overline{s}_4 + \overline{s}_4 s_4 \tag{4.5}$$

and satisfy, in addition to (2.22),

$$[W_2,\,(s_4,\overline{s}_4,W_4)\,]=0 \quad , \quad [W_4,\,s_4]=0=[W_4,\,\overline{s}_4] \quad . \tag{4.6}$$

Notice that W_4 does not commute with s_2 or $\overline{s}_2$ (but, being invariant, it does commute with W_2). As W_2 is proportional to C_2, we only refer to the latter in the sequel. Also, to avoid fractional eigenvalues, we define $W \equiv 4!\,W_4$.

4.2 Decomposition into irreducible representations

In general, a monomial in the c's of ghost number q transforms in $\mathbf{8}^{\wedge q}$, the part of the q–th tensor power of the $su(3)$ adjoint representation that is totally antisymmetric in the q factors. The reduction of the $\mathbf{8}^{\wedge q}$ into irreducible representations of $su(3)$ can be achieved by a variety of methods. One way, which gives results useful in our analysis below, employs conventional tensor methods. We first quote the results

$$\begin{array}{lclcl} \mathbf{8}^{\wedge 0} & = & \mathbf{1} & = & \mathbf{8}^{\wedge 8} \\ \mathbf{8}^{\wedge 1} & = & \mathbf{8} & = & \mathbf{8}^{\wedge 7} \\ \mathbf{8}^{\wedge 2} & = & \mathbf{8}+\mathbf{10}+\overline{\mathbf{10}} & = & \mathbf{8}^{\wedge 6} \\ \mathbf{8}^{\wedge 3} & = & \mathbf{1}+\mathbf{8}+\mathbf{10}+\overline{\mathbf{10}}+\mathbf{27} & = & \mathbf{8}^{\wedge 5} \\ \mathbf{8}^{\wedge 4} & = & 2\times\mathbf{8}+2\times\mathbf{27} & & \end{array} \tag{4.7}$$

noting the symmetry $\mathbf{8}^{\wedge q}=\mathbf{8}^{\wedge(r-q)}$, and then describe the tensorial method of developing the results in fully explicit form.

We may refer to $su(3)$ irreducible representations either by dimension, or else in highest weight $\{\lambda_1,\lambda_2\}$ notation. In the latter notation $\{1,0\}$ and $\{0,1\}$ denote the 'quark' and 'antiquark' representations $\mathbf{3}$ and $\overline{\mathbf{3}}$ each of dimension 3, and $\{\lambda_1,\lambda_2\}$ denotes the representation whose highest weight is $\mathbf{w}(\lambda_1,\lambda_2)=\lambda_1\mathbf{w}(1,0)+\lambda_2\mathbf{w}(0,1)$, where $\mathbf{w}(1,0)$, $\mathbf{w}(0,1)$ are the weights of $\mathbf{3}$, $\overline{\mathbf{3}}$ respectively. The representation $\{\lambda_1,\lambda_2\}$ has dimension

$$d(\lambda_1,\lambda_2)=\frac{1}{2}(\lambda_1+1)(\lambda_2+1)(\lambda_1+\lambda_2+2) \tag{4.8}$$

and Casimir operators (4.1) whose eigenvalues are [43]

$$C_2(\lambda_1,\lambda_2)=\frac{1}{3}(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2)+\lambda_1+\lambda_2 \ . \tag{4.9}$$

$$C_3(\lambda_1,\lambda_2)=\frac{1}{18}(\lambda_1-\lambda_2)(\lambda_1+2\lambda_2+3)(2\lambda_1+\lambda_2+3)=-C_3(\lambda_2,\lambda_1) \ . \tag{4.10}$$

Since $C_3(\lambda_1,\lambda_2)=-C_3(\lambda_2,\lambda_1)$, C_3 vanishes for all self-conjugate ($\lambda_1=\lambda_2$) irreducible representations. The results for the representations ρ that occur in (4.7) are given in the table

$$\begin{array}{ccc} \dim\rho & \{\lambda_1,\lambda_2\} & (C_2,\,C_3) \\ \mathbf{1} & \{0,\,0\} & (0,\,0) \\ \mathbf{8} & \{1,\,1\} & (3,\,0) \\ \mathbf{10} & \{3,\,0\} & (6,\,9) \\ \overline{\mathbf{10}} & \{0,\,3\} & (6,\,-9) \\ \mathbf{27} & \{2,\,2\} & (8,\,0) \end{array} \quad . \tag{4.11}$$

Turning to the tensor analysis of tensors spanned, for $0 \leq q \leq 8$, by the monomials $c_{i_1} \cdots c_{i_q}$, we start with the case $q = 1$, where c_i describes the basis of the $su(3)$ adjoint representation, *i.e.*, an octet. In the case $q = 2$,

$$d_i = f_{ijk} c_j c_k \tag{4.12}$$

describes an independent octet, the only one available since $d_{ijk} c_j c_k \equiv 0$. The remaining tensor, irreducible over the field $\mathbb{R}$, is

$$c_i c_j - \frac{1}{3} f_{ijk} d_k = (\mathbf{20}_2)_{ij} \quad , \tag{4.13}$$

for which $f_{ijk} (\mathbf{20}_2)_{ij} = 0$ by construction. The notation implies it has 20 components, agreeing with the simple count $\binom{8}{2} - 8$. To reduce it into separate **10** and $\overline{\mathbf{10}}$ pieces can be done only over the field of complex numbers, but this is not needed here [44]. We may also write

$$\begin{aligned} c_i c_j = & \ (c_i c_j - \frac{1}{3} f_{ijk} d_k) + \frac{1}{3} f_{ijk} d_k \\ = & \ (P_{20})_{ij,pq} c_p c_q + (P_8)_{ij,pq} c_p c_q \end{aligned} \tag{4.14}$$

where the projectors are given by

$$\begin{aligned} (P_{20})_{ij,pq} &= \frac{1}{2} (\delta_{ip} \delta_{jq} - \delta_{iq} \delta_{jp}) - \frac{1}{3} f_{ijl} f_{lpq} \\ (P_8)_{ij,pq} &= \frac{1}{3} f_{ijl} f_{lpq} \quad . \end{aligned} \tag{4.15}$$

The projection properties and orthogonality can be checked using well known properties of $su(3)$ f-tensors etc. [23, 38]. Also we have trivially

$$P_{20} + P_8 = U \quad , \quad U_{ij,pq} = \frac{1}{2} (\delta_{ip} \delta_{jq} - \delta_{iq} \delta_{jp}) \tag{4.16}$$

where U is the relevant form of the unit operator in the ghost number $q = 2$ space spanned by the antisymmetric tensors $c_i c_j$. Direct calculations on the explicit form for d_k given by (4.12) and for $(\mathbf{20}_2)_{ij}$ by (4.13) show that these have the C_2 eigenvalues 3 and 6 in (4.11).

The space spanned at $q = 3$ by the tensor components $c_i c_j c_k$ gives rise easily to the singlet (0,0)

$$Y = f_{ijk} c_i c_j c_k = c_i d_j \tag{4.17}$$

and the $su(3)$ octet

$$e_i = d_{ijk} c_j d_k \quad . \tag{4.18}$$

This is the only $q = 3$ octet, since

$$\xi_i = f_{ijk} c_j d_k = 0 \tag{4.19}$$

follows from the definition (4.12) of d_k and the Jacobi identity for the f-tensor. We note however that $\xi_i = 0$ is a set of eight non-empty verifiable identities amongst various trilinears $c_i c_j c_k$. To build other irreducible tensors, it is natural to look at the tensors

$$c_i d_j - c_j d_i \tag{4.20}$$

$$c_i d_j + c_j d_i \tag{4.21}$$

with a priori 28 and 36 components. The former (4.20) is irreducible and defines $(\mathbf{20}_3)_{ij}$ as it stands, because $\xi_i = 0$ yield eight identities automatically satisfied by its components. It is also not hard to check that the C_2 eigenvalue is 6. The latter (4.21) is not irreducible, but by extracting the scalar (4.17) and the octet (4.18), we find the irreducible tensor of 27= 36–1–8 components

$$(\mathbf{27}_3)_{ij} = c_i d_j + c_j d_i - \frac{1}{4}\delta_{ij} Y - \frac{6}{5} d_{ijk} e_k \quad . \tag{4.22}$$

It is easy to see that contracting with δ_{ij} and d_{ijk} gives zero as irreducibility requires. It is hard, needing good selection of $su(3)$ such f- and d-tensor identities as found in [23], to prove that C_2 indeed has eigenvalue 8 for $(\mathbf{27}_3)_{ij}$. We could turn results (4.17), (4.18), (4.20) and (4.22) into the form

$$c_i c_j c_k = \sum_R P^R_{ijk,pqr} c_p c_q c_r \tag{4.23}$$

involving a complete set of orthogonal projectors for $R = 1,\ 8,\ 20$ and 27.

Since the case at $q = 4$ involves repetitions, it is best at this point to review the situation regarding octets. At $q = 1,\ 2,\ 3$, we have

$$c_i \quad , \quad d_i \quad , \quad e_i \tag{4.24}$$

and no others. At $q = 4$, we find

$$f_i = d_{ijk} d_j d_k = \Omega_{i\, i_1 i_2 i_3 i_4} c_{i_1} c_{i_2} c_{i_3} c_{i_4} \quad , \tag{4.25}$$

but $f_{ijk} d_j d_k \equiv 0$. A second octet that can be checked easily to be linearly independent of f_i is $Y c_i$. We may build other $q = 4$ octets, but these will not give anything new, since *e.g.* we can prove the results

$$d_{ijk} c_j e_k = -\frac{2}{3} c_i Y \quad , \qquad f_{ijk} c_j e_k = f_i \quad . \tag{4.26}$$

It is thus now obvious that the complete family of octets can be presented as

$$\begin{array}{cccccccc} q = & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ & c_i & d_i & e_i & f_i & & & \\ & & & & Yc_i & Yd_i & Ye_i & Yf_i \quad . \end{array} \tag{4.27}$$

As a check, we find that Yf_i (*e.g.*) is as expected,

$$Yf_1 \sim c_2c_3c_4c_5c_6c_7c_8 \sim *c_1 \quad . \tag{4.28}$$

An alternative but equivalent treatment would employ certain duals of $f_i,\ e_i,\ d_i,\ c_i$ in place of $Yc_i,\ Yd_i,\ Ye_i, Yf_i$ in (4.27). Of course, for the last case, we have just proved the easy bit of the equivalence. In fact, the use of duals in explicit work is much less convenient than the choice used in (4.27). To indicate this, and to do something instructive in its own right, we make explicit the dual relation of f_i and Yc_i (see also [23, section 8]).

To make contact with a dual to the octet f_i of (4.25) replace $c_{i_1}c_{i_2}c_{i_3}c_{i_4}$ there by $\epsilon_{i_1i_2i_3i_4j_1j_2j_3j_4}c_{j_1}c_{j_2}c_{j_3}c_{j_4}$ to reach

$$p_i = \Omega_{i\,i_1i_2i_3i_4}\epsilon_{i_1i_2i_3i_4j_1j_2j_3j_4}c_{j_1}c_{j_2}c_{j_3}c_{j_4} \quad , \tag{4.29}$$

which clearly belongs to $\mathbf{8}^{\wedge 4}$. To relate p_i to Yc_i, we need the identity

$$\frac{1}{4!}\Omega_{i\,i_1i_2i_3i_4}\epsilon_{i_1i_2i_3i_4j_1j_2j_3j_4} = \frac{2}{\sqrt{3}}\delta_{i[j_1}f_{j_2j_3j_4]} \tag{4.30}$$

in which the divisor 4! on the left is actually matched by one implicit in our definition of square antisymmetrisation brackets on the right. Identity (4.30) allows us to prove

$$p_i = \frac{4!\cdot 2}{\sqrt{3}}c_iY = -\frac{4!\cdot 2}{\sqrt{3}}Yc_i \quad , \tag{4.31}$$

as expected.

The contraction $i = j_1$ of (4.30) gives

$$\frac{1}{4!}\Omega_{i_1i_2i_3i_4i_5}\epsilon_{i_1i_2i_3i_4i_5j_1j_2j_3} = \frac{5}{2\sqrt{3}}f_{j_1j_2j_3} \tag{4.32}$$

which is an evident and easily checked analogue of the result

$$\frac{1}{3!}f_{j_1j_2j_3}\epsilon_{j_1j_2j_3i_1i_2i_3i_4i_5} = -2\sqrt{3}\Omega_{i_1i_2i_3i_4i_5} \tag{4.33}$$

given in [23, eqn. (8.14)]. The latter is a contraction of the more useful identity

$$\frac{1}{3!}f_{ij_1j_2}\epsilon_{j_1j_2i_1i_2i_3i_4i_5i_6} = -4\sqrt{3}\delta_{i[i_1}\Omega_{i_2i_3i_4i_5i_6]} \quad . \tag{4.34}$$

This may be used, as we used (4.30), to reach, *e.g.*, eventually the dual relationship of d_i to Ye_i.

Whilst the above tensorial analysis provides an explicit construction from first principles of all the entries of (4.7) the use of s_2 and s_4 expedites explicit work. For example, since $[s_2, X_i] = 0 = [s_4, X_i]$, s_2 and s_4 also commute with C_2 and C_3. Thus, s_2 (*e.g.*) either raises the ghost number of a tensor by one, leaving its $su(3)$ nature unaltered or else annihilates it. Thus $s_2 c_i \sim d_i$, $s_2 d_i = 0$. Similarly, $s_4 c_i \sim f_i$ and $\overline{s}_2 s_4 f_i \sim e_i$. Likewise,

$$s_2(\mathbf{20}_2)_{ij} = s_2(c_i c_j - \frac{1}{3} f_{ijk} d_k) \sim d_i c_j - c_i d_j = -(\mathbf{20}_3)_{ij} \quad , \tag{4.35}$$

since $s_2 d_i = 0$, which confirms what has been seen to hold above.

Further we might expect $s_2(\mathbf{27}_3)_{ij}$ to yield one of the required $(\mathbf{27}_4)_{ij}$. Indeed $s_2 d_i = 0$, $s_2 Y = 0$ and $s_2 e_k = f_k$ allow us to write

$$s_2(\mathbf{27}_3)_{ij} = d_i d_j - \frac{3}{5} d_{ijk} f_k \equiv (\mathbf{27}_4)_{ij} \quad . \tag{4.36}$$

A second 27-tensor in $\mathbf{8}^{\wedge 4}$ that is linearly independent of $(\mathbf{27}_4)_{ij}$ of (4.36) is suggested immediately by duality arguments. One replaces $c_{i_1} c_{i_2} c_{i_3} c_{i_4}$ in $d_i d_j = f_{i i_1 i_2} f_{j i_3 i_4} c_{i_1} c_{i_2} c_{i_3} c_{i_4}$, etc. by $\epsilon_{i_1 i_2 i_3 i_4 j_1 j_2 j_3 j_4} c_{j_1} c_{j_2} c_{j_3} c_{j_4}$. We thereby reach a tensor $(\mathbf{27}'_4)_{ij}$ which is plainly linearly independent of $(\mathbf{27}_4)_{ij}$. It turns out to be proportional to

$$(\mathbf{27}'_4)_{ij} = c_i e_j + c_j e_i - \frac{4}{5} d_{ijk} Y c_k \quad , \tag{4.37}$$

which can be seen to satisfy $d_{ijl}(27'_4)_{ij} = 0$, using (4.26), as well as $(27'_4)_{ii} = 0$, so that it is irreducible, with 27 components. Further, $(27_5)_{ij}$ can now be written down explicitly by action of s_2 on $(27'_4)_{ij}$. No systematic work on projectors for $q = 4$ has been done.

4.3 The Laplacian W

From the analysis of section 4.2 of the $su(3)$ representations contained in $\mathbf{8}^{\wedge q}$, $0 \leq q \leq 8$, where q is the ghost number, it can be seen that the states φ of the system are labelled by the eigenvalues of the ghost number operator $Q = c^i \pi_i$, $Q\varphi = q\varphi$, and of the $su(3)$ Casimirs C_2 and C_3 that label states within each $su(3)$ representation. Since W commutes with $Q,\ C_2,\ C_3$, we expect it to have well defined eigenvalues on all the states of the system and we might further expect it to be defined as a specific function of $Q,\ C_2,\ C_3$.

Some progress can be made analytically to compute W-eigenvalues. For example, for c_i and d_i given by (4.12) which describe $q = 1,\ q = 2$ octets, we may compute directly from (4.5) the results

$$W c_i = 5 c_i \quad , \quad W d_i = 0 \quad . \tag{4.38}$$

These calculations, the latter already non-trivial, depend, amongst other things, on the identities

$$\begin{aligned} \Omega_{i_1 i_2 i_3 i_4 p} \Omega_{i_1 i_2 i_3 i_4 q} &= 5 \delta_{pq} \quad , \\ \Omega_{i_1 i_2 i_3 a b} \Omega_{i_1 i_2 i_3 p q} &= \frac{1}{2}(\delta_{ap}\delta_{bq} - \delta_{aq}\delta_{bp} + f_{abi} f_{pqi}) \quad , \end{aligned} \tag{4.39}$$

of which only the first follows from the definition of Ω easily. Note also that since W distinguishes between different octets, eqn. (4.38), it cannot be a pure function of the Casimirs: it depends also on Q, which does not belong to the $\mathcal{U}(su(3))$ enveloping algebra.

The results of section 4.1 also allow the minimal polynomials for C_2 and C_3 to be deduced. These are

$$C_2(C_2-3)(C_2-6)(C_2-8)=0 \quad , \tag{4.40}$$

$$C_3(C_3+9)(C_3-9)=0 \quad , \tag{4.41}$$

and the orthogonal projectors on the various eigenspaces for C_2 and C_3 are

$$\begin{aligned} P_0^{(2)} &= -\frac{1}{144}(C_2-3)(C_2-6)(C_2-8) & P_0^{(3)} &= -\frac{1}{81}(C_3+9)(C_3-9) \\ P_3^{(2)} &= \frac{1}{45}C_2(C_2-6)(C_2-8) & P_{-9}^{(3)} &= \frac{1}{162}C_3(C_3-9) \\ P_6^{(2)} &= -\frac{1}{36}C_2(C_2-3)(C_2-8) & P_9^{(3)} &= \frac{1}{162}C_3(C_3+9)\ . \\ P_8^{(2)} &= \frac{1}{80}C_2(C_2-3)(C_2-6)\ ; \end{aligned} \tag{4.42}$$

Further progress by analytic methods soon becomes difficult and we have made use of FORM [39]. This enables us firstly to compute all W-eigenvalues, discussed below, and to find the following identities

$$\begin{aligned} C_2C_3 &= 6C_3 \\ {C_3}^2 &= -\frac{9}{4}C_2(C_2-3)(C_2-8) \\ C_2W &= 3W-\frac{1}{2}C_2(C_2-3)(C_2-8) \\ W^2 &= 5W+\frac{2}{27}{C_3}^2\ . \end{aligned} \tag{4.43}$$

These results allow the recovery of (4.40), (4.41) as a mild check on our procedures, and the deduction of the minimal polynomial of W

$$W(W-5)(W-6)=0 \quad , \tag{4.44}$$

which comprises, as it should, all the eigenvalues of W found in practice. Also the orthogonal projectors onto the eigenspaces of W are

$$P_0^{(W)}=\frac{1}{30}(W-5)(W-6) \quad , \quad P_5^{(W)}=-\frac{1}{5}W(W-6) \quad , \quad P_6^{(W)}=\frac{1}{6}W(W-5) \quad . \tag{4.45}$$

Various useful inferences can be made regarding eigenspaces. For example, alongside the previous result $\ker C_2 \subseteq \ker W$, we have $\ker W \subseteq \ker C_3$. Also,

$$\begin{aligned} P_9^{(3)}+P_{-9}^{(3)} &= P_6^{(2)}=P_6^{(W)} \\ P_0^{(2)}P_0^{(W)} &= P_0^{(2)} \\ P_8^{(2)}P_0^{(W)} &= P_8^{(2)} \\ P_0^{(W)}+P_5^{(W)} &= P_0^{(3)} \quad . \end{aligned} \tag{4.46}$$

So far no explicit expression for W in terms of $Q,\ C_2,\ C_3$ is at hand. The major complication in the pattern of the W-eigenvalues of the $su(3)$ representations in $\mathbf{8}^{\wedge q}$ concerns the octets. For these, the ghost number $q = 1, 4, 4, 7$ octets have eigenvalue $W = 5$ and the $q = 2, 3, 5, 6$ octets $W = 0$. This suggests the use of Lagrangian interpolation to define a function

$$f(q) = \frac{1}{360}[(q-4)^2 - 10(q-7)(q-1)](q-2)(q-3)(q-5)(q-6) \quad , \tag{4.47}$$

which equals 1 at $q = 1, 4, 4, 7$ and 0 at $q = 2, 3, 5, 6$, so that for these values, $f(q)^2 = f(q)$. It is then immediate to see that the formula

$$W = \frac{1}{9}C_2(C_2-6)(C_2-8)f(Q) - \frac{1}{6}C_2(C_2-3)(C_2-8) = 5P_3^{(2)}f(Q) + 6P_6^{(2)} \quad , \tag{4.48}$$

correctly predicts the W-eigenvalues of all states. The last two equations of (4.43) also follow directly, using projector properties, $f(Q)^2 = f(Q)$ and the second equation of (4.43). We should stress here that $f(Q)^2 = f(Q)$ holds only for $q \in \{1, 2, \ldots, 7\}$ whereas the allowed range of values of q is $\{0, 1, \ldots, 8\}$. But this does not matter for (4.48), because although $f(Q)$ is finite $(= -27)$ at $q = 0$ and $q = 8$, $C_2 = 0$ for the $q = 0$ and $q = 8$ states. Finally, FORM confirms that W defined by (4.48), and $W = 4!W_4$ given by (4.5) are equal as operators.

A final remark about the form of (4.47), (4.48) is in order here. Observing that $f(Q) = f(8-Q)$, one is led to write (4.47) in terms of $u := Q(8-Q)$, finding

$$f(Q) = F(u) := \frac{1}{40}(u-6)(u-12)(u-15) \tag{4.49}$$

(a form that can also be directly derived by Lagrangian interpolation). A different approach is to start from the minimal polynomial for u

$$u(u-7)(u-12)(u-15)(u-16) = 0 \tag{4.50}$$

and write down directly a function $\tilde{F}(u)$, with $\tilde{F}(u) = 0$ at $u = 0, 12, 15$ and $\tilde{F}(u) = 1$ at $u = 7, 16$

$$\tilde{F}(u) = P_7^{(u)} + P_{16}^{(u)} , \tag{4.51}$$

the projectors $P_\lambda^{(u)}$ being defined in the standard way from (4.50). Using this $\tilde{F}(u)$ in place of $F(u) = f(Q)$ in (4.48) also gives correctly W - the difference $\tilde{F} - F$ is annihilated by $P_3^{(2)}$ [7].

The results of the previous analysis may be summarised in the diagram of figure 4.1 representing the spectra of C_2, C_3 and W. The solid circular disc represents all the $2^8 = 256$ states available in $\bigoplus_{i=0}^{8} \mathbf{8}^{\wedge i}$. The four quadrants represent the four eigenspaces $(0, 3, 6, 8)$

[7] We note incidentally that the operators $M_{ij} := c_i\pi_j - c_j\pi_i$, $i < j$, generate the algebra $spin(8)$, the quadratic Casimir of which is proportional to u (since $M_{ij}M_{ij} = -2u$), *i.e.* (4.48) gives W in terms of the quadratic Casimirs of $su(3)$ and $spin(8)$.

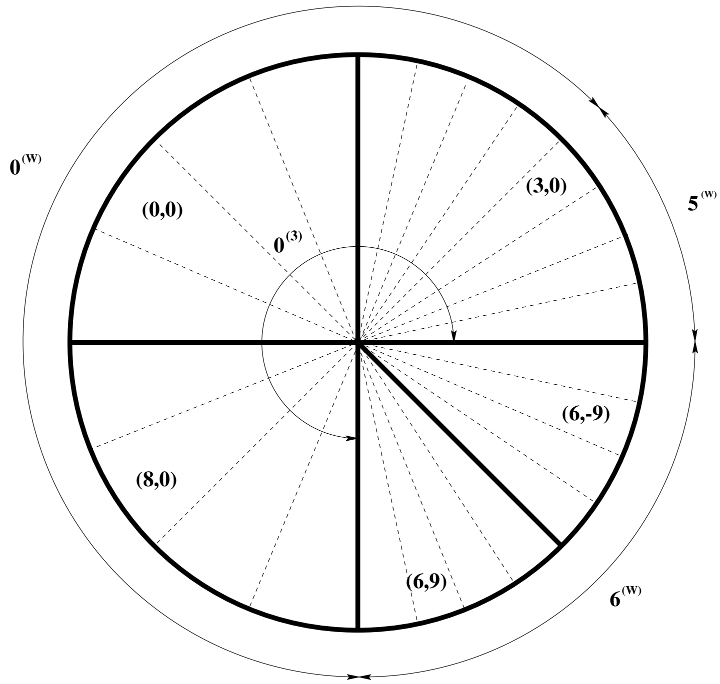

Figure 4.1: The spectrum of the Casimirs (C_2, C_3) and of $W_4 \propto W$

of C_2 while the numbers in parentheses are the eigenvalues of C_2, C_3 valid in each disc segment (bordered by solid black lines). Dashed lines within each disc segment separate multiple copies of the same irreducible representation, corresponding in general to states of different ghost number. The arcs outside the disc specify the three (0,5,6) W–eigenspaces. We summarise its key features:

- The eigenspace $\operatorname{Im} P_6^{(2)}$, equal to $\operatorname{Im} P_6^{(W)}$, is split into two parts with the same number of states, labelled by the C_3 eigenvalues 9 and -9. With the help of (4.11) we recognise these, respectively, as the $\mathbf{10}$ and $\overline{\mathbf{10}}$ $su(3)$ representations, each of which appears four times, with ghost numbers 2, 3, 5 and 6.

- The (0, 0)–subspace (*i.e.* $\ker C_2 = \mathcal{K}_2$), contains four invariant states (the $su(3)$–singlets $\mathbf{1}$), with ghost numbers 0, 3, 5 and 8. All of them are W–harmonic as well, *i.e.*, Σ_4 singlets.

- (8, 0) ($= \mathbf{27}$) appears four times, with ghost numbers 3, 5 and 4 (twice) and is also W–harmonic.

- (3, 0) ($= \mathbf{8}$) appears eight times, with ghost numbers 1 through 7 (twice for 4).

The diagram shows that $\mathcal{K}_2 + \operatorname{Im} P_8^{(2)} \subset \mathcal{K}_4 \subset \ker C_3$. $\mathcal{K}_4$ contains half of the copies of the adjoint representation, the rest belonging to the W–eigenvalue 5.

To look now at the representations of Σ_2 and Σ_4 in (2.22) and (3.15) it is convenient to depict the $su(3)$ representations as in the diagram of figure 4.2 and to analyse there the role played by s_2, s_4 and their adjoints $\overline{s}_2$, $\overline{s}_4$ in interconnecting them. Each *straight line* segment in this diagram represents an irreducible representation. All segments in the same line correspond to the same ghost number, while singlets are represented by circles. The arrows between segments depict the action of the s's (solid (black) lines for s_2, $\overline{s}_2$, dotted (grey) lines for s_4, $\overline{s}_4$); the number below each segment is the W–eigenvalue of the irreducible representation. In the following we denote *e.g.* by $\mathbf{10}_3$ the irreducible representation $\mathbf{10}$ with ghost number 3 while a further superscript u (l) (for *upper* (*lower*)) distinguishes between the two ghost number 4 representations $\mathbf{8}_4$'s and $\mathbf{27}_4$'s. We point out the following

- Referring to multiplets of the superalgebra (3.15), quartets–turned–into–pairs–of–doublets, according to the remark of section 3.3, appear three times. For Σ_2, the quartet $\{\mathbf{8}_4^u, \mathbf{8}_5, \mathbf{8}_3, \mathbf{8}_4^l\}$ actually consists of the pair of doublets $\{\mathbf{8}_5, \mathbf{8}_4^u\}$, $\{\mathbf{8}_4^l, \mathbf{8}_3\}$ —a similar pattern is exhibited by the $\mathbf{27}$'s in the same orders as well as by the Σ_4–quartet $\{\mathbf{8}_4^u, \mathbf{8}_7, \mathbf{8}_1, \mathbf{8}_4^l\}$. The degeneracy seen in the $q = 4$ line is then resolved by noting that s_2 annihilates one octet and $\overline{s}_2$ the other (and similarly for the $\mathbf{27}$).

- Besides the above 'split quartets', we also have the Σ_2–doublets $\{\mathbf{8}_2\,,\,\mathbf{8}_1\}$, $\{\mathbf{10}_3\,,\,\mathbf{10}_2\}$, $\{\overline{\mathbf{10}}_3\,,\,\overline{\mathbf{10}}_2\}$, the Σ_4–doublets $\{\mathbf{10}_5\,,\,\mathbf{10}_2\}$, $\{\overline{\mathbf{10}}_5\,,\,\overline{\mathbf{10}}_2\}$, and their $*$–images $\{\mathbf{8}_7\,,\,\mathbf{8}_6\}$, $\{\mathbf{10}_6\,,\,\mathbf{10}_5\}$, $\{\overline{\mathbf{10}}_6\,,\,\overline{\mathbf{10}}_5\}$ and $\{\mathbf{10}_6\,,\mathbf{10}_3\}$, $\{\overline{\mathbf{10}}_6\,,\,\overline{\mathbf{10}}_3\}$ respectively. Notice that W changes eigenvalue within all Σ_2 doublets involving $\mathbf{8}$'s, reflecting its failure to commute with s_2.

- The $su(3)$ (and hence Σ_2, Σ_4) singlet $\mathbf{1}_0$ is simply the constant monomial 1, while $\mathbf{1}_3$ is the three-cocycle $f_{ijk}c^i c^j c^k = Y$. The other two singlets are the 'top form' $c_1 \ldots c_8$ and the five-cocycle $\Omega_{i_1 \ldots i_5} c^{i_1} \ldots c^{i_5}$, $*$–images of the first two respectively.

- $\mathbf{8}_1$ consists of the 8 c^k's. This is 'lifted' by s_2 to give $\mathbf{8}_2 \sim \{f_{ij}{}^k c^i c^j\}$ and by s_4, giving $\mathbf{8}_4^l \sim \{\Omega_{i_1 i_2 i_3 i_4}{}^k c^{i_1} c^{i_2} c^{i_3} c^{i_4}\}$. $\mathbf{8}_3$ is the image of $\mathbf{8}_1$ under $\overline{s}_2 s_4$, *i.e.* $\mathbf{8}_3 \sim \{f_{i_1 ab} \Omega_{i_2 i_3}{}^{abk} c^{i_1} c^{i_2} c^{i_3}\}$. The $q \to r - q$ symmetry accounts for the rest of the $\mathbf{8}$'s. Notice that $\mathbf{8}_2$, $\mathbf{8}_3$ cannot be lifted by s_4 since they are W–harmonic.

5 Concluding remarks

We have introduced and studied in this paper the supersymmetry algebra generated by the higher order BRST operators. The central term in the algebra is given, in the standard (lowest order) case, by the (quadratic) Casimir. As shown explicitly by the expression of W_4 for the algebra $\mathcal{G} = su(3)$, the higher order Laplacians may involve the ghost number operator, which, unlike the Casimir-Racah operators, lies outside the enveloping algebra $\mathcal{U}(\mathcal{G})$. Thus, the fact that $\Delta \in \mathcal{U}(\mathcal{G})$ in the standard case is rather exceptional.

We wish to conclude with a purely mathematical remark. Using the correspondence $c^i \leftrightarrow$LI forms on the group manifold, the standard BRST operator s_2 may be identified with the exterior derivative d acting on forms. The basic properties of d, $d : \wedge^q \to \wedge^{q+1}$, $d^2 = 0$

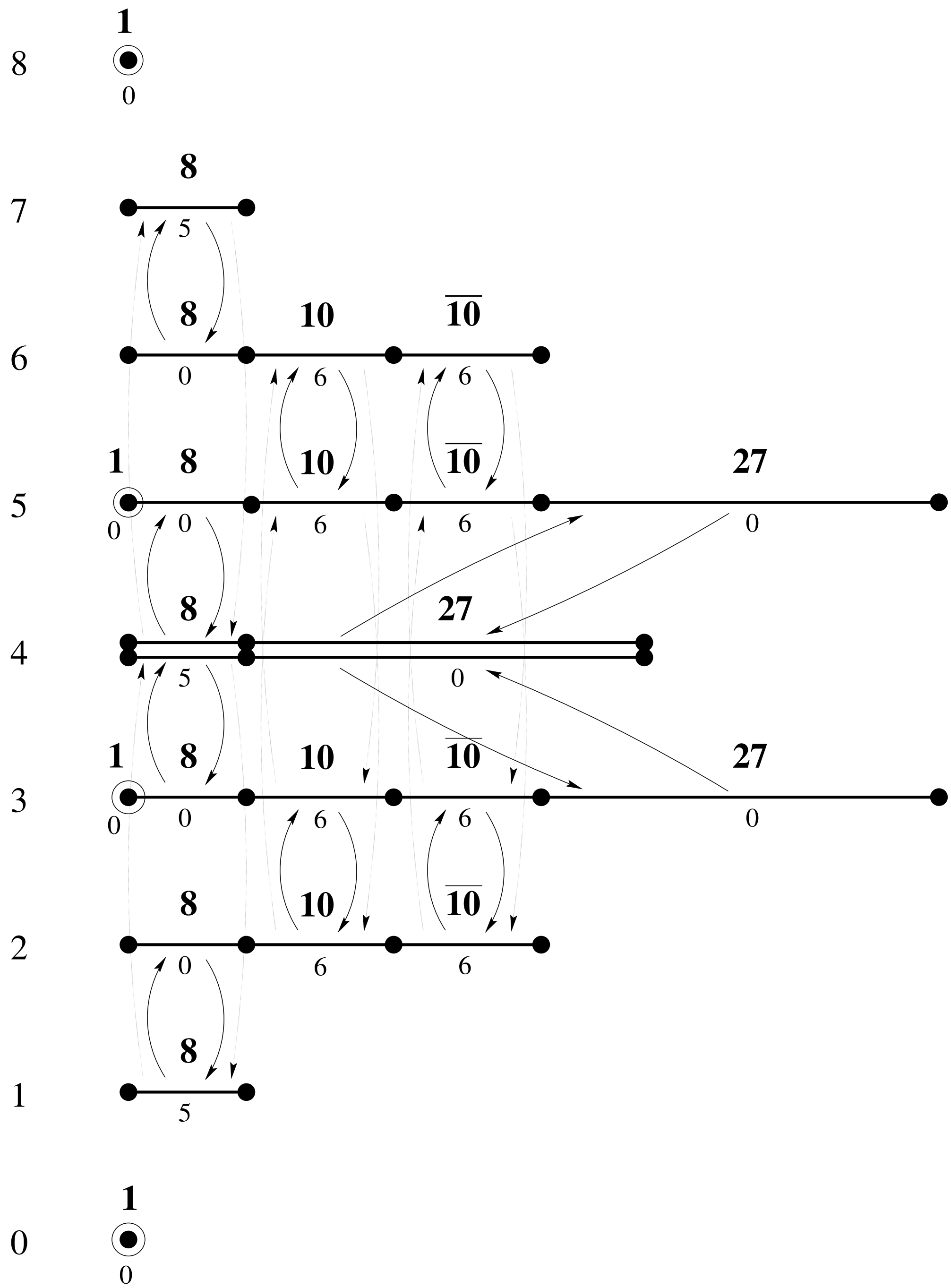

Figure 4.2: Decomposition of $\bigoplus_{i=0}^{8} \mathbf{8}^{\wedge i}$ into irreps of $su(3)$ and the action of s_2 ($\overline{s}_2$) and s_4 ($\overline{s}_4$) on them (notice that $[s_2\,(\overline{s}_2), W] \neq 0$). The eigenvalue of W appears under each irrep.

(and of the codifferential δ) may be extended by introducing generalised operators $\tilde{d}$ in two different ways. One is by replacing the exterior differential by a higher order nilpotent endomorphism $\tilde{d}'$ satisfying $(\tilde{d}')^k = 0$, to study the associated generalised homology, etc. [40, 41]. This approach is reminiscent of the one used to generalise ordinary supersymmetry to fractional supersymmetry (for a review with earlier references see [42]). The second one replaces d by a p-th order differential, $\tilde{d}$, p odd, satisfying $\tilde{d}_p : \wedge^q \to \wedge^{q+p}$, $\tilde{d}_p^2 = 0$, and it is this second point of view which corresponds to the analysis presented in this paper. In fact, the higher BRST operator s_{2m-3} may be considered as an explicit construction of this differential for p=$(2m-3)$, which acts *on LI* forms on the group manifold by translating (3.8) using the above correspondence. $\tilde{d}_p$ is an odd operator satisfying

$$\tilde{d}_p(\alpha \wedge \beta) = (\tilde{d}_p \alpha) \wedge \beta + (-1)^n \alpha \wedge (\tilde{d}_p \beta) \tag{5.1}$$

where n is the order of the LI form α. We recall that, using the (standard) product between manifolds and forms (given by $\langle \mathcal{M}, \alpha \rangle = \int_{\mathcal{M}} \alpha$) one can define the adjoint ∂ of the exterior derivative d. Acting on manifolds, ∂ reduces their dimension by one, is nilpotent, and admits the interpretation as a boundary operator. Using an analogous procedure, one might think of defining the adjoint $\tilde{\partial}_p$ of $\tilde{d}_p$ as an operator. Acting on manifolds it would reduce their dimension by p, being also nilpotent, and the question would arise whether it, too, admits a simple topological interpretation. One might also ask further, whether an analogue of Stokes' theorem could be formulated along these lines or whether the spectrum of the higher order Laplacians studied here provides topological information about the underlying manifold. We do not know whether these mathematical constructions involving $\tilde{d}_p$ can be carried through in general.

To conclude we would like to stress that the cohomological properties used in this paper are also relevant in other related fields, although it may not be directly apparent. They determine and classify, for instance, the local conserved charges in principal chiral models (see [45] and references therein), and are also important in W-algebras (see, for instance, [46, 47, 48] and [49]), where BRST-type techniques, and hence Lie algebra cohomology, are relevant.

Acknowledgements

This research is partially supported by a research grant PB96-0756 from the DGICYT, Spain. C.C. wishes to thank the Spanish Ministry of Education and Culture for a post-doctoral fellowship and J.C.P.B. wishes to thank the Spanish MEC and CSIC for an FPI grant. A.J.M wishes to thank the University of Valencia for hospitality. The research of A.J.M. is supported in part by a grant from PPARC, U.K..

References

[1] C. Becchi, A. Rouet and R. Stora, *Renormalization and gauge theories*, Ann. Phys. (N. Y.) **98**, 287–321 (1976).

[2] I. V. Tyutin, *Gauge invariance in field theory and in statistical mechanics* (1975), Lebedev preprint FIAN#39 (unpublished).

[3] M. Henneaux and C. Teitelboim, *Quantization of gauge systems* (Princeton, 1992).

[4] J. Gomis, J. Paris and S. Samuel, *Antibracket, antifields and gauge-theory quantization*, Phys. Rep. **259**, 1–145 (1995).

[5] E. Witten, *A note on the antibracket formalism*, Mod. Phys. Lett. **A5**, 487–494 (1990).

[6] O. M. Khurdaverian and A. P. Nersessian, *On the geometry of the Batalin-Vilkovisky formalism*, Mod. Phys. Lett. **A8**, 2377–2385 (1993).

[7] M. Alexandrov, M. Kontsevich, A. Schwarz and O. Zaboronsky, *The geometry of the master equation and topological quantum field theory*, Int. J. Mod. Phys. **A12**, 1405–1430 (1997).

[8] A. Nersessian, *Antibracket and non-abelian equivariant cohomology*, Mod. Phys. Lett. **A10**, 3043–3050 (1995).

[9] F. Brandt, M. Henneaux and A. Wilch, *Extended antifield formalism*, Nucl. Phys. **B510**, 640–656 (1998).

[10] L. Álvarez-Gaumé and L. Baulieu, *The two quantum symmetries associated with a classical symmetry*, Nucl. Phys. **B212**, 255–267 (1983).

[11] D. McMullan, *Use of ghosts in the description of physical states*, Phys. Rev. **D33**, 2501–2503 (1986).

[12] J.-L. Gervais, *Group theoretic approach to the string field theory action*, Nucl. Phys. **B276**, 339–348 (1986).

[13] I. Bars and S. Yankielowicz, *Becchi-Rouet-Stora-Tyutin symmetry, differential geometry, and string field theory*, Phys. Rev. **D12**, 3878–3889 (1987).

[14] J. W. van Holten, *The BRST complex and the cohomology of compact Lie algebras*, Nucl. Phys. **B339**, 158–176 (1990).

[15] L. Baulieu, *On the symmetries of topological quantum field theories*, Int. J. Mod. Phys. **A50**, 4483–4500 (1995).

[16] H.S. Yang and B.-H. Lee, *Lie algebra cohomology and group structure of gauge theories*, J. Math. Phys. **37**, 6106–6120 (1996)

[17] T. Kawai, *Remarks on a class of BRST operators*, Phys. Lett. **168B**, 355–359 (1986).

[18] J. M. Figueroa-O'Farrill and T. Kimura, *Some remarks on BRST cohomology, Kähler algebra and 'no-ghost' theorems* (1998), ITP-SP-88-34, unpublished.

[19] L. Bonora and P. Cotta-Ramusino, *Some remarks on BRST transformations, anomalies and the cohomology of the Lie algebra of the group gauge transformations*, Commun. Math. Phys. **87**, 589–603 (1983).

[20] G. Racah, *Sulla caratterizzazione delle rappresentazioni irreducibili dei gruppi semisimplici di Lie*, Lincei-Rend. Sc. fis. mat. e nat. **VIII**, 108–112 (1950), Princeton lectures, CERN-61-8 (reprinted in Ergeb. Exact Naturwiss. **37**, 28-84 (1965), Springer-Verlag).

[21] I. M. Gel'fand, *The center of an infinitesimal group ring*, Mat. Sbornik **26**, 103–112 (1950), (English trans. by Los Alamos Sci. Lab., AEC-TR-6133 (1963)).

[22] A. M. Perelomov and V. S. Popov, *Casimir operators for semisimple groups*, Math. USSR-Izvestija **2**, 1313–1335 (1968).

[23] J. A. de Azcárraga, A. J. Macfarlane, A. J. Mountain and J. C. Pérez Bueno, *Invariant tensors for simple groups*, Nucl. Phys. **B510**, 657–687 (1998).

[24] J. A. de Azcárraga and J. C. Pérez Bueno, *Higher-order simple Lie algebras*, Commun. Math. Phys. **184**, 669–681 (1997).

[25] T. Lada and J. Stasheff, *Introduction to SH Lie algebras for physicists*, Int. J. Theor. Phys. **32**, 1087–1103 (1993).

[26] E. Witten and B. Zwiebach, *Algebraic structures and differential geometry in two-dimensional string theory*, Nucl. Phys. **B377**, 55–112 (1992).

[27] B. H. Lian and G. J. Zuckerman, *New perspectives on the BRST-algebraic structure of string theory*, Commun. Math. Phys. **154**, 613–646 (1993).

[28] K. Bering, P. H. Damgaard and J. Alfaro, *Algebra of higher antibrackets*, Nucl. Phys. **B478**, 459–503 (1996).

[29] F. Akman, *On some generalizations of Batalin-Villkovisky algebras*, J. Pure Appl. Algebra **120**, 105–141 (1997).

[30] J. Stasheff, *The (secret?) homological algebra of the Batalin-Vilkovisky approach*, hep-th/9712157, to appear in *Proceedings of the Conference on Secondary Calculus and Cohomological Physics*, Moscow, August 1997.

[31] J. A. de Azcárraga, J. M. Izquierdo and J. C. Pérez Bueno, *An Introduction to some novel applications of Lie algebra cohomology in mathematics and physics*, physics/9803046, to appear in the Proceedings of the VI Fall Workshop on Geometry and Physics (Salamanca, September 1997).

[32] C. Chevalley and S. Eilenberg, *Cohomology theory of Lie groups and Lie algebras*, Trans. Am. Math. Soc. **63**, 85–124 (1948).

[33] J. A. de Azcárraga and J. M. Izquierdo, *Lie groups, Lie algebras, cohomology and some applications in physics* (Camb. Univ. Press, 1995).

[34] F. A. Berezin, *Differential forms on supermanifolds*, Sov. J. Nucl. Phys. **30**, 605–609 (1979); *Introduction to superanalysis* (Reidel, 1987).

[35] W. V. D. Hodge, *The theory and applications of harmonic integrals* (Camb. Univ. Press, 1941).

[36] H. Samelson, *Topology of Lie groups*, Bull. Am. Math. Soc. **57**, 2–37 (1952).

[37] W. Greub, S. Halperin and R. Vanstone, *Connections, Curvature and Cohomology*, vol. III (Acad. Press, 1976).

[38] A. J. Macfarlane, A. Sudbery and P. H. Weisz, *On Gell-Mann's λ matrices, d- and f-tensors, octets, and parametrisations of $SU(3)$*, Commun. Math. Phys. **11**, 77 (1968).

[39] J. A. M. Vermaseren, *Symbolic Manipulation with FORM version 2: Tutorial and Reference Manual* (CAN, 1992).

[40] M. Dubois-Violette, *Generalized spaces with $d^N = 0$ and the q-differential calculus*, Czech. J. Phys. **46**, 1227–1233 (1997).

[41] R. Kerner, *Z_n–graded differential calculus*, Czech. J. Phys. **47**, 33–40 (1997).

[42] R. S. Dunne, A. J. Macfarlane, J. A. de Azcárraga and J. C. Pérez Bueno, *Geometrical foundations of fractional supersymmetry*, Int. J. Mod. Phys. **A19**, 3275–3305 (1997).

[43] L.C. Biedenharn, Phys. Lett. **3**, 69-70 (1963).

[44] A. Sudbery, PhD Thesis, Cambridge University, 1969.

[45] J.M. Evans, M. Hassan, N.J. MacKay and A.J. Mountain, *Local conserved charges and principal chiral models*, hep-th/9902008

[46] F.A. Bais, P. Bouwknegt, M. Surridge and K. Schoutens, *Extensions of the Virasoro algebra constructed from Kac-Moody algebras using higher order Casimir invariants*, Nucl. Phys. **B304**, 348–370 (1988)

[47] E. Bergshoeff, A. Bilal and K.S. Stelle, *W-symmetries: gauging and symmetry*, Mod. Phys. Lett. **A6**, 4959–4984 (1991)

[48] H. Lu, C.N. Pope, X.J. Wang and K.W. Xu, *The complete cohomology of the W_3 string*, Class. Quantum Grav. **11**, 967-981 (1994)

[49] P. Bouwknegt, J. McCarty, K. Pilch, *The W_3 algebra*, Springer-Verlag (1996)