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The Role of the Canonical Element in the Quantized Algebra of Differential Operators $\mathcal{A}\rtimes\mathcal{U}$ [1]

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Abstract

We review the construction of the cross product algebra $\mathcal{A}\rtimes\mathcal{U}$ from two dually paired Hopf algebras $\mathcal{U}$ and $\mathcal{A}$. The canonical element in $\mathcal{U}\otimes\mathcal{A}$ is then introduced, and its properties examined. We find that it is useful for giving coactions on $\mathcal{A}\rtimes\mathcal{U}$, and it allows the construction of objects with specific invariance properties under these coactions. A "vacuum operator" is found which projects elements of $\mathcal{A}\rtimes\mathcal{U}$ onto said objects. We then discuss bicovariant vector fields in the context of the canonical element.

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1 Introduction

Given two dually paired Hopf algebras $\mathcal{U}$ and $\mathcal{A}$, it is well known that a new algebra, the cross product algebra $\mathcal{A}\rtimes\mathcal{U}$, can be constructed, and that this algebra may be viewed as consisting of differential operators and the functions they act on. In Section 2, we review the standard way of constructing this algebra [1, 2, 3]. The interpretation of this algebra as one of *bicovariant* objects requires the introduction of well-defined actions and coactions on $\mathcal{A}\rtimes\mathcal{U}$; these are given in Section 3, again in the standard fashion [3].

However, the emphasis of this paper will be on exploring the properties of the canonical element C of $\mathcal{U}\otimes\mathcal{A}$, which enters very naturally through the expressions for the coactions, as is shown in Section 4. Furthermore, we can recover many of the familiar relations for quantum groups from the consistency relations which C satisfies in the case where $\mathcal{U}$ is quasitriangular [4, 5]. In Section 5, we find new right and left invariant elements of $\mathcal{A}\rtimes\mathcal{U}$, as well as an element which realizes the vacuum operator in $\mathcal{A}\rtimes\mathcal{U}$. Viewed geometrically, this object projects vector fields and functions onto their value at the identity. Finally, in Section 6, we present a method for constructing bicovariant vector fields in $\mathcal{U}$ with particularly simple coactions, and examine its relation to the canonical element.

Much of the notation used in this paper may be found in [2, 4] and references therein.

2 Hopf Algebras and their Duals, and the Cross Product

We begin with a review of the basic relations for a Hopf algebra, and the process by which its dual is also made a Hopf algebra.

Let $\mathcal{U}$ be a Hopf algebra, *i.e.* $\mathcal{U}$ is an associative algebra over a field k with unit $1_{\mathcal{U}}$, and the coproduct $\Delta : \mathcal{U} \to \mathcal{U}\otimes\mathcal{U}$, antipode $S : \mathcal{U} \to \mathcal{U}$, and counit $\epsilon : \mathcal{U} \to k$ are all linear maps satisfying the usual relations [2]. Let $\mathcal{A}$ be the dual of $\mathcal{U}$, *i.e.* the set of linear functionals of elements of $\mathcal{U}$. $\mathcal{A}$ is a vector space in the usual manner, and can be endowed with a Hopf algebra structure via the definitions

$$\langle x, ab\rangle \equiv \langle \Delta(x), a\otimes b\rangle \, ,$$

$$
\begin{aligned}
\langle x, 1_{\mathcal{A}} \rangle &\equiv \epsilon(x), \\
\langle x \otimes y, \Delta(a) \rangle &\equiv \langle xy, a \rangle , \\
\langle x, S(a) \rangle &\equiv \langle S(x), a \rangle , \\
\epsilon(a) &\equiv \langle 1_{\mathcal{U}}, a \rangle ,
\end{aligned} \tag{1}
$$

($x, y \in \mathcal{U}$, $a, b \in \mathcal{A}$) which give, respectively, product, unit, coproduct, antipode, and counit on $\mathcal{A}$. $\mathcal{U}$ and $\mathcal{A}$ are said to be *dually paired*. We may now introduce a linearly independent basis $\{e_i\}$ for $\mathcal{U}$, *i.e.* $\mathcal{U} = \mathrm{span}\{e_i | i \in \mathcal{J}\}$ where $\mathcal{J}$ is the appropriate index set, and take the basis of $\mathcal{A}$ to be $\{f^i\}$ given by

$$
\left\langle e_i, f^j \right\rangle = \delta_i^j. \tag{2}
$$

We will use these bases extensively throughout the remainder of this paper.

Now that we have two dually paired Hopf algebras, we can introduce a unital algebra which is denoted $\mathcal{A}\rtimes\mathcal{U}$, the "cross product" of $\mathcal{A}$ and $\mathcal{U}$. (This construction is also called the "smash product", and is a Hopf algebra generalization of the Heisenberg double and the Weyl semidirect product.) $\mathcal{A}\rtimes\mathcal{U}$ is constructed to be isomorphic to $\mathcal{A} \otimes \mathcal{U}$ as a vector space. This may be seen explicitly through the definition of the multiplication on $\mathcal{A}\rtimes\mathcal{U}$:

$$
\begin{aligned}
a \cdot b &\equiv ab \otimes 1_{\mathcal{U}}, \\
x \cdot y &\equiv 1_{\mathcal{A}} \otimes xy, \\
a \cdot x &\equiv a \otimes x, \\
x \cdot a &\equiv a_{(1)} \otimes x_{(2)} \left\langle x_{(1)}, a_{(2)} \right\rangle ,
\end{aligned} \tag{3}
$$

where $a, b \in \mathcal{A}$, $x, y \in \mathcal{U}$, and we use the notation introduced by Sweedler [1]:

$$
\Delta(\alpha) = \alpha_{(1)} \otimes \alpha_{(2)}. \tag{4}
$$

Note that this multiplication is associative, and also that $\mathcal{A}\rtimes\mathcal{U}$ contains subalgebras isomorphic to both $\mathcal{A}$ and $\mathcal{U}$. Therefore, we shall drop the "$\cdot$" from now on, assuming that the above multiplication is used when we take the product of an element of $\mathcal{A}$ with an element of $\mathcal{U}$. Furthermore, we shall also write e_i and f^i rather than $1_{\mathcal{A}} \otimes e_i$ and $f^i \otimes 1_{\mathcal{U}}$ respectively when viewing the bases of $\mathcal{U}$ and $\mathcal{A}$ as elements in $\mathcal{A}\rtimes\mathcal{U}$.

Notice that although $\mathcal{U}$ and $\mathcal{A}$ are both Hopf algebras, $\mathcal{A}\rtimes\mathcal{U}$ is not, *i.e.* $\mathcal{A}\rtimes\mathcal{U}$ is an algebra that does not admit a coproduct or counit (and therefore no antipode) even though the subalgebras $\mathcal{U}$ and $\mathcal{A}$ do. Despite this fact, the original Hopf algebraic structures of $\mathcal{U}$ and $\mathcal{A}$ are preserved.

3 Actions and Coactions

Suppose we have an algebra $\mathcal{B}$ and a vector space $\mathcal{V}$; a *left action* of $\mathcal{B}$ on $\mathcal{V}$ is a bilinear map $\triangleright : \mathcal{B} \otimes \mathcal{V} \to \mathcal{V}$ satisfying

$$(xy) \triangleright v = x \triangleright (y \triangleright v) \tag{5}$$

for all $x, y \in \mathcal{B}$ and $v \in \mathcal{V}$. A right action $\triangleleft$ of $\mathcal{B}$ on $\mathcal{V}$ can be defined similarly. In the case where $\mathcal{B}$ is a Hopf algebra and $\mathcal{V}$ is a unital algebra, we further require that for $x \in \mathcal{B}$ and $a, b \in \mathcal{V}$,

$$\begin{aligned} x \triangleright (ab) &= (x_{(1)} \triangleright a)(x_{(2)} \triangleright b), \\ 1_{\mathcal{B}} \triangleright a &= a \\ x \triangleright 1_{\mathcal{V}} &= 1_{\mathcal{V}} \epsilon(x). \end{aligned} \tag{6}$$

In this case, $\triangleright$ is called a *generalized derivation*, and we can interpret $\mathcal{B}$ as an algebra of differential operators which act on functions (*i.e.* elements of $\mathcal{V}$).

Now, suppose we have a coalgebra $\mathcal{C}$ and a vector space $\mathcal{V}$; a *left coaction* of $\mathcal{C}$ on $\mathcal{V}$ is a linear map ${}_{\mathcal{C}}\Delta : \mathcal{V} \to \mathcal{C} \otimes \mathcal{V}$ satisfying

$$\begin{aligned} (\Delta \otimes \mathrm{id})_{\mathcal{C}}\Delta(v) &= (\mathrm{id} \otimes_{\mathcal{C}}\Delta)_{\mathcal{C}}\Delta(v), \\ (\epsilon \otimes \mathrm{id})_{\mathcal{C}}\Delta(v) &= v, \end{aligned} \tag{7}$$

for all $v \in \mathcal{V}$, where Δ and ϵ are the coproduct and counit on $\mathcal{C}$, respectively. The right coaction $\Delta_{\mathcal{C}}$ is defined similarly. If $\mathcal{C}$ is a Hopf algebra and $\mathcal{V}$ is a unital algebra, we impose the further conditions that

$$\begin{aligned} {}_{\mathcal{C}}\Delta(ab) &= {}_{\mathcal{C}}\Delta(a)_{\mathcal{C}}\Delta(b), \\ {}_{\mathcal{C}}\Delta(1_{\mathcal{V}}) &= 1_{\mathcal{C}} \otimes 1_{\mathcal{V}}, \end{aligned} \tag{8}$$

for $a, b \in \mathcal{V}$, *i.e.* ${}_{\mathcal{C}}\Delta$ must be an algebra homomorphism.

We now introduce specific actions and coactions in the case where we have the two Hopf algebras $\mathcal{U}$ and $\mathcal{A}$ and the associative unital algebra $\mathcal{A} \rtimes \mathcal{U}$. The left and right actions of $\mathcal{U}$ on $\mathcal{A} \rtimes \mathcal{U}$ are defined as

$$\begin{aligned} x \triangleright \alpha &\equiv x_{(1)} \alpha S(x_{(2)}), \\ \alpha \triangleleft x &\equiv S(x_{(1)}) \alpha x_{(2)}, \end{aligned} \tag{9}$$

for $x \in \mathcal{U}$, $\alpha \in \mathcal{A}{\rtimes}\mathcal{U}$. Note that for the case where $\alpha \in \mathcal{U}$, these are the left and right adjoint actions, and when $\alpha \in \mathcal{A}$, the right action of x on α may be written

$$x \triangleright \alpha = \alpha_{(1)} \left\langle x, \alpha_{(2)} \right\rangle, \tag{10}$$

which is the usual action of a differential operator x on a function α.

Keeping in mind that the coaction should describe the transformation properties of the elements of $\mathcal{A}{\rtimes}\mathcal{U}$, we make the following choices: $\mathcal{A}$ left coacts on $\mathcal{A}{\rtimes}\mathcal{U}$ so as to leave $\mathcal{U}$ invariant, *i.e.*

$${}_{\mathcal{A}}\Delta(x) \equiv 1_{\mathcal{A}} \otimes x, \tag{11}$$

$x \in \mathcal{U}$. Furthermore, $\mathcal{A}$ left and right coacts on $\mathcal{A}$ via the coproduct:

$${}_{\mathcal{A}}\Delta(a) = \Delta_{\mathcal{A}}(a) = \Delta(a), \tag{12}$$

$a \in \mathcal{A}$. The right coaction of $\mathcal{A}$ on $\mathcal{U}$ is a bit more complicated; we first use a Sweedler-like notation to write the coactions as

$${}_{\mathcal{A}}\Delta(\alpha) \equiv \alpha^{(1')} \otimes \alpha^{(2)}, \qquad \Delta_{\mathcal{A}}(\alpha) \equiv \alpha^{(1)} \otimes \alpha^{(2')}, \tag{13}$$

where the unprimed elements live in $\mathcal{A}{\rtimes}\mathcal{U}$, the primed ones in $\mathcal{A}$. Inspired by the form of (10), we therefore define the two pieces of $\Delta_{\mathcal{A}}(x)$ for $x \in \mathcal{U}$ to be those quantities which satisfy

$$y \triangleright x \equiv x^{(1)} \left\langle y, x^{(2')} \right\rangle \tag{14}$$

for $y \in \mathcal{U}$. This may look a bit mysterious, and one might wonder if this really defines both $x^{(1)}$ and $x^{(2')}$. To see that it actually does, we use our dual basis to write $\Delta_{\mathcal{A}}(x)$ as

$$\Delta_{\mathcal{A}}(x) \equiv x_i \otimes f^i, \tag{15}$$

where $x_i \in \mathcal{U}$. (Throughout this paper, repeated indices are summed over the index set $\mathcal{J}$.) Therefore,

$$e_j \triangleright x = x_i \left\langle e_j, f^i \right\rangle = x_j, \tag{16}$$

giving

$$\Delta_{\mathcal{A}}(x) = (e_i \triangleright x) \otimes f^i. \tag{17}$$

All of the above definitions are consistent with the conditions necessary for $\Delta_{\mathcal{A}}$ to be a right coaction.

Similarly, we can define a left coaction of $\mathcal{U}$ on $\mathcal{A}\rtimes\mathcal{U}$:

$$ {}_{\mathcal{U}}\Delta : \mathcal{A}\rtimes\mathcal{U} \to \mathcal{U}\otimes\mathcal{A}\rtimes\mathcal{U}; \quad \mathcal{A}\rtimes\mathcal{U} \ni \alpha \mapsto {}_{\mathcal{U}}\Delta(\alpha) \equiv \alpha^{(\bar{1})}\otimes\alpha^{(2)}. \tag{18} $$

On $\mathcal{U}$, ${}_{\mathcal{U}}\Delta$ is the coproduct:

$$ {}_{\mathcal{U}}\Delta(x) \equiv x^{(\bar{1})}\otimes x^{(2)} = x_{(1)}\otimes x_{(2)} \equiv \Delta(x), \quad x\in\mathcal{U}. \tag{19} $$

On $\mathcal{A}$, ${}_{\mathcal{U}}\Delta$ is defined again implicitly via

$$ ab = b_{(1)}\langle a^{(\bar{1})}, b_{(2)}\rangle a^{(2)}, \qquad a,b\in\mathcal{A}. \tag{20} $$

Using the right action of a function b on another function a given by

$$ a \triangleleft b \equiv S(b_{(1)})ab_{(2)}, \tag{21} $$

one can easily show that

$$ {}_{\mathcal{U}}\Delta(a) = e_i\otimes(a\triangleleft f^i). \tag{22} $$

In the following sections we focus on the nontrivial coactions $\Delta_{\mathcal{A}}$ and ${}_{\mathcal{U}}\Delta$ which, for simplicity, we refer to as the right and left coactions respectively. For example, an element α of $\mathcal{A}\rtimes\mathcal{U}$ will be called "left invariant" if ${}_{\mathcal{U}}\Delta(\alpha) = 1_{\mathcal{U}}\otimes\alpha$, while "right invariant" elements satisfy $\Delta_{\mathcal{A}}(\alpha) = \alpha\otimes 1_{\mathcal{A}}$.

4 The Canonical Element

We now introduce the canonical element C, which lives in $\mathcal{U}\otimes\mathcal{A}$, and has the form

$$ C \equiv e_i\otimes f^i. \tag{23} $$

C satisfies several relations; for instance, note that

$$ \begin{aligned} (\Delta\otimes\mathrm{id})(C) &= \Delta(e_i)\otimes f^i \\ &= (e_i)_{(1)}\otimes(e_i)_{(2)}\otimes f^i \\ &= e_i\otimes e_j\otimes f^i f^j \\ &= (e_i\otimes 1_{\mathcal{U}}\otimes f^i)(1_{\mathcal{U}}\otimes e_j\otimes f^j) \\ &= C_{13}C_{23} \end{aligned} \tag{24} $$

(where in going from the second to the third line we have used the duality between $\mathcal{U}$-comultiplication and $\mathcal{A}$-multiplication). Similar calculations also give $(\mathrm{id}\otimes\Delta)(C) = C_{12}C_{13}$, as well as the following:

$$\begin{aligned}(S\otimes\mathrm{id})(C) = (\mathrm{id}\otimes S)(C) &= C^{-1},\\ (\epsilon\otimes\mathrm{id})(C) = (\mathrm{id}\otimes\epsilon)(C) &= 1_{\mathcal{U}}\otimes 1_{\mathcal{A}}.\end{aligned} \tag{25}$$

C does more than just satisfying the above relations; to see that this is true, we can compute the right coaction of a basis vector in $\mathcal{U}$. Using (17),

$$\begin{aligned}\Delta_{\mathcal{A}}(e_i) &= (e_j \triangleright e_i)\otimes f^j\\ &= (e_j)_{(1)} e_i S((e_j)_{(2)})\otimes f^j\\ &= e_m e_i S(e_n)\otimes f^m f^n\\ &= (e_m\otimes f^m)(e_i\otimes 1_{\mathcal{A}})(S(e_n)\otimes f^n)\\ &= C(e_i\otimes 1_{\mathcal{A}})(S\otimes\mathrm{id})(C),\end{aligned} \tag{26}$$

so for any $x\in\mathcal{U}$,

$$\Delta_{\mathcal{A}}(x) = C(x\otimes 1_{\mathcal{A}})C^{-1}. \tag{27}$$

However, when we think of C as living in $(\mathcal{A}\rtimes\mathcal{U})\otimes(\mathcal{A}\rtimes\mathcal{U})$, with e_i and f^i as the bases for the subalgebras $\mathcal{U}$ and $\mathcal{A}$ of $\mathcal{A}\rtimes\mathcal{U}$ respectively, further results follow; for instance, if $a\in\mathcal{A}$,

$$\begin{aligned}C(a\otimes 1_{\mathcal{A}\rtimes\mathcal{U}})C^{-1} &= e_i a S(e_j)\otimes f^i f^j\\ &= (a_{(1)}(e_i)_{(2)}\left\langle (e_i)_{(1)}, a_{(2)}\right\rangle)S(e_j)\otimes f^i f^j\\ &= a_{(1)}\left\langle (e_k)_{(1)}, a_{(2)}\right\rangle (e_k)_{(2)} S((e_k)_{(3)})\otimes f^k\\ &= a_{(1)}\otimes\left\langle e_k, a_{(2)}\right\rangle f^k\\ &= a_{(1)}\otimes a_{(2)},\end{aligned} \tag{28}$$

(where $1_{\mathcal{A}\rtimes\mathcal{U}}\equiv 1_{\mathcal{A}}\otimes 1_{\mathcal{U}}$) so that

$$C(a\otimes 1_{\mathcal{A}\rtimes\mathcal{U}})C^{-1} = \Delta(a). \tag{29}$$

Thus, the right coaction of $\mathcal{A}$ on $\mathcal{A}\rtimes\mathcal{U}$ can be written as

$$\Delta_{\mathcal{A}}(\alpha) = C(\alpha\otimes 1_{\mathcal{A}\rtimes\mathcal{U}})C^{-1} \tag{30}$$

for any $\alpha \in \mathcal{A} \rtimes \mathcal{U}$. (This expression shows explicitly that $\Delta_{\mathcal{A}}$ is an algebra homomorphism.) We can continue doing calculations along these lines, and we find

$$C^{-1}(1_{\mathcal{A} \rtimes \mathcal{U}} \otimes \alpha) C = {}_{\mathcal{U}}\Delta(\alpha) \tag{31}$$

for $\alpha \in \mathcal{A} \rtimes \mathcal{U}$ (so that, for $x \in \mathcal{U}$, $\Delta(x) = C^{-1}(1_{\mathcal{A} \rtimes \mathcal{U}} \otimes x) C)$. Using these results, together with the coproduct relations for C, we obtain the equation

$$C_{23} C_{12} = C_{12} C_{13} C_{23}. \tag{32}$$

(Interestingly, this equation can be viewed as giving the multiplication on $\mathcal{A} \rtimes \mathcal{U}$ as defined in (3).)

In the case where $\mathcal{U}$ is a quasitriangular Hopf algebra with universal R-matrix $\mathcal{R}$, the coproduct relations involving C imply the following consistency conditions:

$$\begin{aligned} \mathcal{R}_{12} C_{13} C_{23} &= C_{23} C_{13} \mathcal{R}_{12}, \\ \mathcal{R}_{23} C_{12} &= C_{12} \mathcal{R}_{13} \mathcal{R}_{23}, \\ \mathcal{R}_{13} C_{23} &= C_{23} \mathcal{R}_{13} \mathcal{R}_{12}. \end{aligned} \tag{33}$$

To see the added significance of these equations, note that

$$\langle C, a \otimes \mathrm{id} \rangle = a, \tag{34}$$

where $a \in \mathcal{A}$, and we use the convention

$$\langle \alpha, \mathrm{id} \rangle = \alpha \tag{35}$$

for any $\alpha \in \mathcal{A} \rtimes \mathcal{U}$. Let $\rho : \mathcal{U} \to M_n(k)$ be an $n \times n$ matrix representation of $\mathcal{U}$, and define the n^2 matrix elements $A^i{}_j \in \mathcal{A}$ by

$$\left\langle x, A^i{}_j \right\rangle \equiv \rho^i{}_j(x). \tag{36}$$

(These $A^i{}_j$s are what are usually viewed as the noncommuting matrix elements of the pseudomatrix group associated with $\mathcal{U}$ [6].) Given ρ, we can define the $\mathcal{U}$-valued matrices $L^{\pm}$ by

$$\begin{aligned} L^{+} &\equiv (\mathrm{id} \otimes \rho)(\mathcal{R}), \\ L^{-} &\equiv (\rho \otimes \mathrm{id})(\mathcal{R}^{-1}), \end{aligned} \tag{37}$$

and the numerical R-matrix by

$$R \equiv (\rho \otimes \rho)(\mathcal{R}). \tag{38}$$

Furthermore, it is easily seen that $(\rho \otimes \mathrm{id})(C) = A$. Now let us apply $(\rho^i{}_k \otimes \rho^j{}_l \otimes \mathrm{id})$ to the first of equations (33); the left hand side gives

$$\begin{aligned}(\rho^i{}_k \otimes \rho^j{}_l \otimes \mathrm{id})(\mathcal{R}_{12}C_{13}C_{23}) &= (\rho^i{}_m \otimes \rho^j{}_n)(\mathcal{R})(\rho^m{}_k \otimes \mathrm{id})(C)(\rho^n{}_l \otimes \mathrm{id})(C) \\ &= R^{ij}{}_{mn}A^m{}_kA^n{}_l.\end{aligned} \tag{39}$$

The right hand side gives $A^i{}_mA^j{}_nR^{mn}{}_{kl}$, so using the usual notation, we obtain

$$RA_1A_2 = A_2A_1R, \tag{40}$$

which gives the commutation relations between the elements of A. Doing similar gymnastics with the other two equations in (33) gives

$$\begin{aligned}L_1^+A_2 &= A_2R_{21}L_1^+, \\ L_1^-A_2 &= A_2R^{-1}L_1^-,\end{aligned} \tag{41}$$

which give the commutation relations between elements of $\mathcal{U}$ and $\mathcal{A}$ within $\mathcal{A}\rtimes\mathcal{U}$. (Of course, we also have the commutation relations

$$\begin{aligned}RL_2^\pm L_1^\pm &= L_1^\pm L_2^\pm R, \\ RL_2^+L_1^- &= L_1^-L_2^+R,\end{aligned} \tag{42}$$

between elements of $\mathcal{U}$, obtained as above from $\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$, the quantum Yang-Baxter equation.) Thus, we recover all the commutation relations between A and $L^\pm$ given in [5]. In this case, $\mathcal{A}$ is interpreted as the function algebra on the quantum group generated by the elements of A, and $\mathcal{U}$ as the universal enveloping algebra generated by elements of $L^\pm$.

5 The Vacuum Operator

An arbitrary element α of $\mathcal{A}\rtimes\mathcal{U}$ will, in general, have nontrivial left and right coactions (as defined at the end of Section 3). We can, however, associate to each such element one left invariant and one right invariant element of $\mathcal{A}\rtimes\mathcal{U}$ (denoted $\overrightarrow{\alpha}$ and $\overleftarrow{\alpha}$ respectively), given by the formulae

$$\overrightarrow{\alpha} = \alpha^{(2)}S^{-1}(\alpha^{(\bar{1})}), \qquad \overleftarrow{\alpha} = S^{-1}(\alpha^{(2')})\alpha^{(1)}. \tag{43}$$

One may easily check that ${}_{\mathcal{U}}\Delta(\vec{\alpha}) = 1_{\mathcal{U}} \otimes \vec{\alpha}$ and $\Delta_{\mathcal{A}}(\overleftarrow{\alpha}) = \overleftarrow{\alpha} \otimes 1_{\mathcal{A}}$ for $\alpha \in \mathcal{A} \rtimes \mathcal{U}$.

Notice that for $x \in \mathcal{U}$,

$$\vec{x} = x^{(2)} S^{-1}(x^{(\bar{1})}) = x_{(2)} S^{-1}(x_{(1)}) = \epsilon(x) 1_{\mathcal{U}}, \tag{44}$$

and similarly, for $a \in \mathcal{A}$,

$$\overleftarrow{a} = S^{-1}(a_{(2)}) a_{(1)} = \epsilon(a) 1_{\mathcal{A}}. \tag{45}$$

Recall now the definitions of left and right vacua [5], denoted by $\Omega_{\mathcal{A}}$ and $\Omega_{\mathcal{U}}$ respectively, which satisfy

$$\begin{aligned} L^+\Omega_{\mathcal{U}} = L^-\Omega_{\mathcal{U}} = I\Omega_{\mathcal{U}}, \qquad \Omega_{\mathcal{A}} A = \Omega_{\mathcal{A}} I, \\ \langle \Omega_{\mathcal{A}} | \Omega_{\mathcal{U}} \rangle \;=\; 1, \end{aligned} \tag{46}$$

where I is the unit matrix. These have, so far, been introduced "by hand". We now show that they can be related to a special element of $\mathcal{A} \rtimes \mathcal{U}$. Consider the element E of $\mathcal{A} \rtimes \mathcal{U}$, constructed out of C as follows:

$$E = m_{\mathcal{A} \rtimes \mathcal{U}} \circ (S^{-1} \otimes \mathrm{id}) \circ \tau(C) = S^{-1}(f^i) e_i \tag{47}$$

(with $m_{\mathcal{A} \rtimes \mathcal{U}}$ the multiplication on $\mathcal{A} \rtimes \mathcal{U}$ and $\tau(\alpha \otimes \beta) = \beta \otimes \alpha$ for all $\alpha,\ \beta \in \mathcal{A} \rtimes \mathcal{U}$). We easily find that $E^2 = E$. Furthermore, for $a \in \mathcal{A}$,

$$\begin{aligned} Ea &= S^{-1}(f^i) e_i a \\ &= S^{-1}(f^i) a_{(1)} \left\langle (e_i)_{(1)}, a_{(2)} \right\rangle (e_i)_{(2)} \\ &= S^{-1}(f^i f^j) a_{(1)} \left\langle e_i, a_{(2)} \right\rangle e_j \\ &= S^{-1}(f^j) S^{-1}(a_{(2)}) a_{(1)} e_j \\ &= E\epsilon(a), \end{aligned} \tag{48}$$

and similarly, for $x \in \mathcal{U}$,

$$xE = \epsilon(x) E. \tag{49}$$

Note that (48) and (49) imply that

$$ExaE = \langle x, a \rangle E \text{ and } EaxE = \epsilon(x)\epsilon(a) E. \tag{50}$$

Provided we generalize (46) so that for $x \in \mathcal{U}$, $a \in \mathcal{A}$,

$$\begin{aligned} x\Omega_{\mathcal{U}} = \Omega_{\mathcal{U}} x &= \epsilon(x)\Omega_{\mathcal{U}}, \\ \Omega_{\mathcal{A}} a = a\Omega_{\mathcal{A}} &= \Omega_{\mathcal{A}}\epsilon(a), \end{aligned} \tag{51}$$

(which give as a consequence $\langle \Omega_{\mathcal{A}} | x a \, \Omega_{\mathcal{U}} \rangle = \langle x, a \rangle$), the preceding properties of E suggest a representation given by

$$E \simeq |\Omega_{\mathcal{U}}\rangle \langle \Omega_{\mathcal{A}}| \, , \tag{52}$$

so that "vacuum operator" is perhaps an appropriate name for E.
Remark: There also exists an object $\bar{E}$, given by

$$\bar{E} = m_{\mathcal{A} \rtimes \mathcal{U}} \circ (S^2 \otimes \mathrm{id})(C) \equiv S^2(e_i) f^i \tag{53}$$

which has properties similar to that of E, *e.g.* $\bar{E}^2 = \bar{E}$, $\bar{E}x = \bar{E}\epsilon(x)$ and $a\bar{E} = \epsilon(a)\bar{E}$ for $x \in \mathcal{U}$, $a \in \mathcal{A}$; thus, $\bar{E} \simeq |\Omega_{\mathcal{A}}\rangle \langle \Omega_{\mathcal{U}}|$ is the vacuum operator "adjoint" to E.

The element E, however, has more properties. For example, for $x \in \mathcal{U}$,

$$\begin{aligned} E \overleftarrow{x} &= E S^{-1}(x^{(2')}) x^{(1)} \\ &= E \epsilon(x^{(2')}) x^{(1)} \\ &= E x \end{aligned} \tag{54}$$

(where we have used $(\mathrm{id} \otimes \epsilon) \circ \Delta_{\mathcal{A}} = \mathrm{id}$). Similarly, we find for $a \in \mathcal{A}$,

$$aE = \overrightarrow{a} \, E. \tag{55}$$

With the help of (44) and (45), we can summarize:

$$\alpha E = \overrightarrow{\alpha} \, E, \quad E\alpha = E \, \overleftarrow{\alpha}, \quad \alpha \in \mathcal{A} \rtimes \mathcal{U}. \tag{56}$$

It is interesting to note that, in the classical limit, E becomes a formal Taylor expansion operator; *e.g.* E applied to a function $f(x)$ on a one-dimensional space returns $f(0) = f(x - x)$ [7]. A more detailed exposition of these matters will appear in a forthcoming paper [8].

6 Bicovariant Vector Fields

The appearance of a (possibly) infinite sum in equation (17), or for that matter (30), suggests that the elements of $\mathcal{U}$ have in general very complicated transformation properties. In contrast, the elements of $\mathcal{A}$, especially those constructed from the matrix entries of A, have very simple transformation properties given by the coproduct in $\mathcal{A}$ (12). We would like to show how to construct vector fields corresponding to — and inheriting the simple behavior of — these functions. This construction can then be used to find a basis for vector fields that closes under coaction and hence under (mutual) adjoint actions. First we need to prove the following lemma:

Lemma: *Let* $\Upsilon \equiv \Upsilon_i \otimes \Upsilon^i \in \mathcal{U} \otimes \mathcal{U}$ *be such that* $\Upsilon\Delta(x) = \Delta(x)\Upsilon$ *for all* $x \in \mathcal{U}$*; it then follows that* $\Upsilon_i \otimes (x \triangleright \Upsilon^i) = (\Upsilon_i \triangleleft x) \otimes \Upsilon^i$ *with* $\Upsilon_i \triangleleft x \equiv S(x_{(1)})\Upsilon_i x_{(2)}$ *for all* $x \in \mathcal{U}$*.*

Proof:

$$\begin{aligned} \Upsilon_i \otimes (x \triangleright \Upsilon^i) &\equiv \Upsilon_i \otimes x_{(1)}\Upsilon^i S(x_{(2)}) \\ &= S(x_{(1)})x_{(2)}\Upsilon_i \otimes x_{(3)}\Upsilon^i S(x_{(4)}) \\ &= S(x_{(1)})\Upsilon_i x_{(2)} \otimes \Upsilon^i x_{(3)} S(x_{(4)}) \\ &= (\Upsilon_i \triangleleft x) \otimes \Upsilon^i. \ \Box \end{aligned} \tag{57}$$

For any function $b \in \mathcal{A}$, define

$$Y_b \equiv \langle \Upsilon, b \otimes \mathrm{id} \rangle \ \in \mathcal{U}. \tag{58}$$

This vector field has the following transformation property:

$$\Delta_{\mathcal{A}}(Y_b) = Y_{b_{(2)}} \otimes S(b_{(1)})b_{(3)} \tag{59}$$

Proof:

$$\begin{aligned} \Delta_{\mathcal{A}}(Y_b) &= \langle \Upsilon_i, b \rangle \, (e_k \triangleright \Upsilon^i) \otimes f^k \\ &= \langle \Upsilon_i \triangleleft e_k, b \rangle \, \Upsilon^i \otimes f^k \\ &= \left\langle \Upsilon_i \otimes e_k, b_{(2)} \otimes S(b_{(1)})b_{(3)} \right\rangle \Upsilon^i \otimes f^k \\ &= Y_{b_{(2)}} \otimes S(b_{(1)})b_{(3)}. \ \Box \end{aligned} \tag{60}$$

Example: Let $\Upsilon \equiv \mathcal{R}_{21}\mathcal{R}_{12}$ and $b \equiv A^i{}_j$; then $Y^i{}_j \equiv Y_{A^i{}_j} = \langle \mathcal{R}_{21}\mathcal{R}_{12}, A^i{}_j \otimes id \rangle$ is the well-known matrix of vector fields $L^+S(L^-)$ introduced in [9] with coaction $\Delta_{\mathcal{A}}(Y^i{}_j) = Y^k{}_l \otimes S(A^i{}_k)A^l{}_j$.

This last example may in some cases provide a way of computing the canonical element C from $\mathcal{R}_{21}\mathcal{R}_{12}$: let μ be the map

$$\mu : \mathcal{A} \to \mathcal{U}, \ b \mapsto \langle \mathcal{R}_{21}\mathcal{R}_{12}, b \otimes \mathrm{id} \rangle \, . \tag{61}$$

There is a certain class of quasitriangular Hopf algebras for which this map is invertible (*i.e.* the factorizable quasitriangular Hopf algebras); therefore, if $\mathcal{U}$ is factorizable, the fact that

$$\begin{aligned} (\mathrm{id} \otimes \mu)(C) &= e_i \left\langle \mathcal{R}_{21}\mathcal{R}_{12}, f^i \otimes \mathrm{id} \right\rangle \\ &= \mathcal{R}_{21}\mathcal{R}_{12} \end{aligned} \tag{62}$$

implies that we can find an explicit form for C:

$$C = (\mathrm{id} \otimes \mu^{-1})(\mathcal{R}_{21}\mathcal{R}_{12}). \tag{63}$$

7 Conclusion

In this letter we have shown how the cross product algebra $\mathcal{A} \rtimes \mathcal{U}$ (interpreted as a quantized algebra of differential operators and functions) constructed from the two dual Hopf algebras $\mathcal{A}$ and $\mathcal{U}$ admits transformations (*i.e.* coactions) of its elements given through conjugation by the *canonical element* C of $\mathcal{U} \otimes \mathcal{A} \subseteq (\mathcal{A} \rtimes \mathcal{U}) \otimes (\mathcal{A} \rtimes \mathcal{U})$. We therefore have, in principle, a way of finding the coactions on the differential operators and functions, although the actual computations using C may be difficult. Since the coactions on $\mathcal{A} \rtimes \mathcal{U}$ can be expressed as inner automorphisms, they are manifestly homomorphic, and they explicitly preserve the commutation relations (3).

We have also found a general way of constructing objects in $\mathcal{A} \rtimes \mathcal{U}$ with particular invariance properties, *e.g.* elements $\vec{\alpha}$ which are invariant under the left coaction of $\mathcal{U}$. In doing so, we note the existence of the vacuum operator E, which projects out the left or right invariant parts of elements of $\mathcal{A} \rtimes \mathcal{U}$.

The question of possibly finding C more specifically than in terms of the bases of $\mathcal{U}$ and $\mathcal{A}$ depends on the invertibility of the map μ; unfortunately, there does not seem to be any a priori reason to assume that μ^{-1} exists in the nonfactorizable case [10], so the general expression for C may be as good as we can get.

As a final comment, we note that the canonical element is somewhat reminiscent of the universal R-matrix constructed via the Drinfel'd quantum double [11]. However, the similarity is only superficial; for instance, C does not satisfy a Yang-Baxter equation, but instead satisfies (32). Even so, this similarity suggests the possibility of axiomatizing the definition of a cross product algebra in the same way as we define a quasitriangular Hopf algebra, by postulating the existence of an associative unital algebra $\mathcal{P}$, together with an element $C \in \mathcal{P} \otimes \mathcal{P}$ which satisfies (32). A cross product algebra would then be a specific case of this, in analogy to the fact that a quantum double is a particular type of quasitriangular Hopf algebra.

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