

COPRODUCT OF VECTOR FIELDS ON THE QUANTUM LORENTZ GROUP*

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We use properties of the $L^\pm$ matrices, specific to the case of the q -deformed Lorentz group, to derive expressions for the coproduct of its bivariant vector fields.

1. Introduction

Among the plethora of q -deformations that recent years have witnessed, that of the Lorentz group^{1,2} stands perhaps as the most tangible link between quantum groups and physics. From this (physical) point of view, the exploration of the corresponding dual structure, i.e., the Lorentz algebra, is of particular interest as it should provide the basic ingredients for, among other things, the q -deformation of relativistic quantum field theory. The Lorentz algebra has been studied in Refs. 3 and 4 (a generalization of the results given there to the case of other groups appears in Ref. 5). What we consider here is an alternative, simpler way to derive one of the results of Ref. 4, namely the expression for the coproduct of the vector fields on the group.

We begin by briefly reviewing the formalism for complexified quantum groups developed in Refs. 3 and 4 as it applies to the case of $SL_q(2, \mathbb{C})$. Introduce the 2×2 matrix t of functions on the group whose elements, members of a non-commutative algebra A , satisfy the relations⁶ ($q > 0$ real):

$$\tilde{R} t_1 t_2 = t_1 t_2 \tilde{R}, \quad \det_q t \equiv t_{11} t_{22} - q t_{12} t_{21} = 1. \quad (1)$$

Here $t_1 \equiv t \otimes I$ (with $\otimes$ denoting matrix tensor product: $(A \otimes B)_{ij,kl} \equiv A_{ik} B_{jl}$), $t_2 \equiv I \otimes t$ and $\tilde{R}$ is the symmetric R -matrix for this group, suitably normalized:

$$\tilde{R} = q^{-1/2} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

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The first of (1) determines the commutation relations among the elements t_{ij} of t while the second is the q -analog of a unit-determinant condition (one easily checks that this is consistent with the commutation relations, i.e., that $\det_q t$ is indeed central).

Complex conjugation of (1) gives rise to four more generators t_{ij}^* whose algebra is most conveniently expressed in terms of $\hat{t}_{ij} \equiv (S(t_{ij}))^*$ (here $S(\cdot)$ denotes the antipode map and $S(t_{ij}) = (t^{-1})_{ij}$). One finds:

$$\hat{R}^{-1} \hat{t}_1 \hat{t}_2 = \hat{t}_1 \hat{t}_2 \hat{R}^{-1}, \quad \det_q \hat{t} \equiv \hat{t}_{11} \hat{t}_{22} - q \hat{t}_{12} \hat{t}_{21} = 1. \quad (2)$$

Finally, consistent commutation relations between t and $\hat{t}$ must be given. These are

$$\hat{R} \hat{t}_1 t_2 = t_1 \hat{t}_2 \hat{R}. \quad (3)$$

All the commutation relations presented so far can be encoded in a single RTT equation:

$$\hat{R} T_1 T_2 = T_1 T_2 \hat{R} \quad (4)$$

with the definitions:

$$T = \begin{pmatrix} t & 0 \\ 0 & \hat{t} \end{pmatrix}, \quad \hat{R} = \begin{pmatrix} \hat{R} & 0 & 0 & 0 \\ 0 & 0 & \hat{R} & 0 \\ 0 & \hat{R}^{-1} & 0 & 0 \\ 0 & 0 & 0 & \hat{R}^{-1} \end{pmatrix}.$$

In the following we will use frequently the notation³: $T_{ij} \equiv t_{ij}$, $\hat{T}_{ij} \equiv \hat{t}_{ij}$ (while $T_{ij} = T_{ji} = 0$) $i, j, \hat{i}, \hat{j} = 1, 2$.

Having completed the description of the algebra A of functions on the group and their complex conjugates, we now turn our attention to its dual, A^* . To this end, one introduces the matrices of linear functionals $L^\pm$ in the form³:

$$L^+ = \begin{pmatrix} L_{11}^+ & L_{12}^+ & 0 & 0 \\ 0 & L_{22}^+ & 0 & 0 \\ 0 & 0 & L_{11}^+ & 0 \\ 0 & 0 & L_{21}^+ & L_{22}^+ \end{pmatrix}, \quad L^- = \begin{pmatrix} L_{11}^- & L_{12}^- & 0 & 0 \\ L_{21}^- & L_{22}^- & 0 & 0 \\ 0 & 0 & L_{11}^- & L_{12}^- \\ 0 & 0 & L_{21}^- & L_{22}^- \end{pmatrix}. \quad (5)$$

Their elements operate on those of T according to

$$(L_{IJ}^\pm, T_{KL}) = (\hat{R}^{\pm 1})_{LJ}^{IK}, \quad (I) = (i, \hat{i}) \quad i, \hat{i} = 1, 2 \quad (6)$$

while $\langle L_{IJ}^\pm, 1 \rangle = \delta_{IJ}$. We will also need the standard formulas³ (summation over repeated indices implied):

$$\Delta(L_{IJ}^\pm) = L_{IK}^\pm \otimes L_{KJ}^\pm, \quad S(L_{IJ}^\pm) = ((L^\pm)^{-1})_{IJ} \quad (7)$$

for the coproduct and the antipode of $L^\pm$ respectively.

We close this review of the formalism with the definition of the object under study here, i.e., the matrix Y of linear functionals given by

$$Y \equiv \begin{pmatrix} y & 0 \\ 0 & \hat{y} \end{pmatrix} \equiv L^+ S(L^-). \quad (8)$$

For the motivation behind this definition see Ref. 7.

2. Coproduct of Y

The task of expressing $\Delta(Y_{IJ})$ in terms of the Y 's, rather than the L 's was accomplished in Ref. 4 in two different ways. In the first approach, one starts by expressing $\Delta(Y_{IJ})$ in terms of the L 's using the definition (8) and the known result for $\Delta(L_{MN}^\pm)$, given in (7), and then essentially inverts (8) (to the extend needed) to obtain the desired expression in terms of the Y 's. The second approach is based on a study of the action of Y_{IJ} on monomials in the elements of T and subsequently 'reads off' the desired expressions, using in an intermediate step a set of operators with relatively simple action on such monomials. Both methods involve rather tedious algebra the need for which, as we now show, can be eliminated. Start from (8) to get

$$\begin{aligned} \Delta(Y_{IJ}) &= \Delta(L_{IK}^+) \Delta(S(L_{KJ}^-)) \\ &= (L_{IS}^+ \otimes L_{SK}^+) (S(L_{RJ}^-) \otimes S(L_{KR}^-)) \\ &= L_{IS}^+ S(L_{RJ}^-) \otimes Y_{SR} \quad \text{or,} \\ \Delta(Y_\alpha) &= \mathcal{O}_{\alpha\beta} \otimes Y_\beta \quad \alpha, \beta = 1, \dots, 16 \end{aligned} \quad (9)$$

where $\mathcal{O} \equiv L^+ \otimes S(L^-)^T$ and Y is now a column vector:

$$Y^T = (Y_{11}, Y_{12}, 0, 0, Y_{21}, \dots, Y_{22}).$$

Also, (8) implies

$$Y^{-1} = L^- S(L^+). \quad (10)$$

In Ref. 4, the following relations were shown to hold for $U_q(\mathfrak{sl}(2, \mathbb{C}))$ (no summation implied):

$$L_{ii}^+ L_{ii}^+ = 1, \quad L_{ij}^- = L_{ij}^-, \quad i, j, \hat{i}, \hat{j} = 1, 2. \quad (11)$$

These, together with the known relations⁶: $L_{11}^+ L_{22}^+ = L_{11}^+ L_{22}^+ = 1$ give

$$L_{11}^+ = L_{22}^+, \quad L_{22}^+ = L_{11}^+. \quad (12)$$

Consider now the matrix:

$$B \equiv L^- (L_{11}^+)^{-1} \equiv \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.$$

Use Eqs. (10)–(12) and the specific form of L^+ to compute, as an example, A_{12} :

$$\begin{aligned} A_{12} &\equiv L_{12}^-(L_{11}^+)^{-1} = L_{12}^-(L_{22}^+)^{-1} \\ &= L_{12}^-(L^+)^{-1}_{22} \\ &= L_{12}^-S(L^+)^{-1}_{22} \\ &= (Y^{-1})_{12} \end{aligned}$$

since $S(L^+)^{-1}_{12} = 0$. Proceeding in the same fashion for the rest of its elements, one finds for A :

$$A = \begin{pmatrix} (Y^{-1})_{11} & (Y^{-1})_{12} \\ (Y^{-1})_{21} & (Y^{-1})_{22} \end{pmatrix}$$

which in turn gives $\mathcal{O}$ in terms of the Y 's:

$$\begin{aligned} \mathcal{O} &\equiv L^+ \underline{\otimes} S(L^-)^T \\ &= L^+ S(L^-) L^- (L_{11}^+)^{-1} \underline{\otimes} L_{11}^+ S(L^-)^T \\ &= YB \underline{\otimes} ((L^+)^{-1}_{11})^{-1} S(L^-)^T \\ &= YB \underline{\otimes} S(L_{11}^+)^{-1} S(L^-)^T \\ &= YB \underline{\otimes} S(B)^T. \end{aligned} \tag{13}$$

$$\tag{14}$$

Substituting into (13) explicit expressions for the elements of Y^{-1} , we find after a little algebra (in agreement with the results of Ref. 4):

$$\Delta(y) = \begin{pmatrix} F & y_A F' \\ 0 & y_B F' \end{pmatrix} \otimes y, \quad \Delta(\tilde{y}) = \begin{pmatrix} y_B F' & 0 \\ y_C F' & F \end{pmatrix} \otimes \tilde{y},$$

where

$$y_A \equiv y_+ \tilde{y}_1 - q^{-2} y_1 \tilde{y}_+, \quad y_B \equiv \tilde{y}_1 y_2 - q^2 \tilde{y}_+ y_-, \quad y_C \equiv \tilde{y}_- y_2 - q^2 \tilde{y}_2 y_-,$$

y is a column vector: $y^T = (y_{11}, y_{12}, y_{21}, y_{22}) \equiv (y_1, y_+, y_-, y_2)$ (similarly for $\tilde{y}$),

$$F \equiv \begin{pmatrix} S(y_2) & -q^2 S(y_-) \\ -q^{-2} S(\tilde{y}_+) & S(\tilde{y}_1) \end{pmatrix},$$

and $\otimes$ denotes matrix multiplication and algebraic tensor product. As an example:

$$\Delta(y_1) = S(y_2) \otimes y_1 - q^2 S(y_-) \otimes y_+ + y_A S(y_2) \otimes y_- - q^2 y_A S(y_-) \otimes y_2.$$

Similar considerations can provide the coproduct of the Y 's for other complex quantum groups provided their $L^\pm$ matrices have the form of (5). It remains to be

seen whether relatively simple procedures like the above would be sufficient in the case of higher-dimensional groups.

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