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Physics, combinatorics and Hopf algebras

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A number of problems in theoretical physics share a common nucleus of a combinatoric nature. It is argued here that Hopf algebraic concepts and techiques can be particularly efficient in dealing with such problems. As a first example, a brief review is given of the recent work of Connes, Kreimer and collaborators on the algebraic structure of the process of renormalization in quantum field theory. Then the concept of k-primitive elements is introduced — these are particular linear combinations of products of Feynman diagrams — and it is shown, in the context of a toy-model, that they significantly reduce the computational cost of renormalization. As a second example, Sorkin's proposal for a family of generalizations of quantum mechanics, indexed by an integer $k > 2$, is reviewed (classical mechanics corresponds to $k = 1$, while quantum mechanics to $k = 2$). It is then shown that the quantum measures of order k proposed by Sorkin can also be described as k-primitive elements of the Hopf algebra of functions on an appropriate infinite dimensional abelian group.

Keywords: Hopf algebras; renormalization; primitive elements; generalized quantum mechanics; quantum measures.

Una serie de problemas en física teórica comparte un núcleo común de índole combinatório. Es la tesis de este artículo que conceptos y técnicas de álgebras de Hopf pueden ser particularmente eficientes en el tratamiento de este tipo de problemas. Como un primer ejemplo, se presenta un resumen del trabajo recien de Connes, Kreimer y sus colaboradores sobre la estructura algebráica del proceso de renormalización en teoría cuántica de campos. Despues, se introduce el concepto de elementos k-primitivos — estos son combinaciones lineales particulares de productos de diagramas de Feynman — y se demuestra, en el contexto de un modelo de juguete, que reducen de manera esencial el costo computacional de la renormalización. Como un segundo ejemplo, la propuesta de Sorkin de una familia de generalizaciones de mecánica cuántica, indexada por un entero $k > 2$, es presentada (mecánica clásica corresponde a $k = 1$, mientras mecánica cuántica a $k = 2$). Se muestra en continuación que las medidas cuánticas de orden k propuestas por Sorkin pueden también ser descritas como elementos k-primitivos del álgebra de Hopf de funciones sobre un grupo abeliano de dimensión infinita apropiadamente definido.

Descriptores: Algebras de Hopf; renormalización; elementos primitivos; mecánica cuántica generalizada; medidas cuánticas.

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1. Introduction

This paper deals with cases where combinatoric problems arising in physics may be efficiently handled by geometric means. Two particular examples are used to illustrate the point: the quest for primitive elements in the Hopf algebra of renormalization of Connes and Kreimer [5, 6], on one hand, and the classification of possible generalizations of quantum mechanics, proposed by Sorkin [13, 14], according to the additivity properties of the corresponding "quantum measure", on the other. In both cases, an underlying infinite-dimensional Lie group structure permits the geometrization of the problem, as a result of which combinatoric operations are handled by differential geometric machinery, both facilitating and illuminating its solution. In the first case, the group is the non-abelian group of renormalization schemes, introduced by Connes and Kreimer, while in the second, it is the abelian group of characteristic functions of subsets of the set of histories of a quantum particle, introduced in Ref. 3. The renormalization problem is presented in Sec. 3, the source of the results being [4]. Generalized quantum mechanics appears in Sec. 4, the exposition following [3]. Before that, in Sec. 2, a Hopf algebra primer translates some familiar Lie group concepts into a language suitable for the applications at hand. A final section of conclusions is there to transmit a sense of order.

2. A Hopf Algebra Primer

The amount of Hopf algebraic machinery needed in the sequel is quite modest. In fact, it is no more than what the concept of a Lie group supplies, appropriately dualized. Thus, in principle, everything in this paper could be cast in familiar Lie group language, but at the cost of significant notational inconvenience. I opt for a quick translation of Lie groups into Hopf algebraic terms — the reader will find that it is well worth the modest initial investment.

The definition of a Lie group G entails the concepts of a *product* $\cdot$, a *unit* e, and an *inverse* g^{-1}, together with appropriate smoothness conditions. These are operations defined in terms of the *points* of the group manifold: given two points, one can associate whit them their product, there is a special point which is neutral with respect to multiplication etc. Admitting linear combinations of group elements, with real coefficients, one obtains the *group algebra* $\mathbb{R}(G)$, with the product inherited from that of the group, and defined to distribute over the addition, so that, *e.g.*,

$$(\lambda_1 g_1 + \lambda_2 g_2) \cdot g_3 = \lambda_1 (g_1 \cdot g_3) + \lambda_2 (g_2 \cdot g_3), \qquad \lambda_i \in \mathbb{R}, \qquad g_i \in G.$$

Dual to the vector space generated by the points in G is the vector space $\mathrm{Fun}(G)$ of functions on G. The duality is

via a bilinear inner product $\langle\cdot,\cdot\rangle\colon \mathbb{R}(G)\otimes \text{Fun}(G)\to\mathbb{R}$, given simply by the evaluation of the function at the point, $g\otimes f\mapsto\langle g,f\rangle\equiv f(g)$. Notice that the space of functions is endowed with an algebra structure as well, given by free commutativity. Now, one has learned in quantum mechanics that, given an inner product between two vector spaces and an operator acting on one of them, one can define its adjoint, acting on the other. In an analogous fashion, one may dualize the above defining operations of a Lie group to corresponding operations on the functions on the group. To make this look nice, everything is expressed as maps. The product in G is then a bilinear map $m\colon G\otimes G\to G$, $m(g_1\otimes g_2)\equiv g_1\cdot g_2$ (we may define m on $G\otimes G$, rather than $G\times G$ because of bilinearity). Similarly, the unit is formalized into a map $\eta:\mathbb{R}\to\mathbb{R}(G)$, $\lambda\mapsto\eta(\lambda)=\lambda e$, *i.e.*, η sends every number to that same number times the unit of the group. Finally, the inverse is promoted to the map $S\colon\mathbb{R}(G)\to\mathbb{R}(G)$, $g\mapsto S(g)=g^{-1}$, extended by *linearity* to the entire $\mathbb{R}(G)$. This last feature may sound weird for an inverse map, and indeed it is as it implies, for example, that $S(g_1+g_2)=S(g_1)+S(g_2)=g_1^{-1}+g_2^{-1}$, or, even worse, $S(\lambda g)=\lambda g^{-1}$. To avoid confusion, S is called the *antipode* map, with the property that it returns the group inverse when evaluated on the group elements themselves.

The stage is now set for dualizing everything. One may start with the easiest, the unit map η. Its dual is the *counit* map $\epsilon\colon\text{Fun}(G)\to\mathbb{R}$, $f\mapsto\epsilon(f)=f(e)$, *i.e.* the counit of a function is a number, its value at the identity (notice how dualizing reverses the direction of the arrows in the map definitions). Next comes the dual of the antipode, also called the antipode, and also denoted by S,

$$\langle S(g),f\rangle=\langle g^{-1},f\rangle\equiv\langle g,S(f)\rangle, \tag{1}$$

i.e., the antipode $S(f)$ of a function f, is a function that, when evaluated on a point g, returns the value of f on g^{-1}. Just like its dual, S extends by linearity to the whole of $\text{Fun}(G)$. Finally, dual to the product map is the *coproduct* map

$$\Delta:\text{Fun}(G)\to\text{Fun}(G)\otimes\text{Fun}(G),$$
$$f\mapsto\Delta(f)\equiv\sum_i f^i_{(1)}\otimes f^i_{(2)}\equiv f_{(1)}\otimes f_{(2)},$$

such that

$$\begin{aligned}\langle m(g_1\otimes g_2),f\rangle&=\langle g_1\otimes g_2,\Delta(f)\rangle\\&\equiv\langle g_1\otimes g_2,f_{(1)}\otimes f_{(2)}\rangle\\&\equiv\langle g_1,f_{(1)}\rangle\langle g_2,f_{(2)}\rangle.\end{aligned}\tag{2}$$

Some remarks on the notation used might be helpful. The coproduct of a function f is $\Delta(f)$, a function of *two* arguments, such that $\Delta(f)(g_1,g_2)=f(g_1g_2)$ (I will occasionally drop the dot from the group product). Now, the tensor product of two functions, of a single argument each, can be considered as a function of two arguments, $(f\otimes f')(g_1,g_2)=\langle g_1\otimes g_2,f\otimes f'\rangle=f(g_1)f'(g_2)$, where the last equation *defines* the inner product between $\mathbb{R}(G)\otimes\mathbb{R}(G)$ and $\text{Fun}(G)\otimes\text{Fun}(G)$. A general function of two arguments though, and, in particular, $\Delta(f)$, does not necessarily factorize like this but involves instead a sum over such tensor products, written above as $\sum_i f^i_{(1)}\otimes f^i_{(2)}$ for the case of $\Delta(f)$. To further enhance confusion, one usually drops the summation symbol and the associated index and writes simply $\Delta(f)=f_{(1)}\otimes f_{(2)}$, a powerful notation due to Sweedler.

An example might be needed to illustrate the above. Consider the group $A(1)$ of affine transformations of the real line, *i.e.* of maps $(a,b)\colon x\mapsto ax+b$, with a,b real and $a>0$. The group law is given by composition,

$$(a,b)\cdot(a',b')\equiv(a,b)\circ(a',b')=(aa',ab'+b)\,, \tag{3}$$

with the unit being the point $(1,0)$ and the inverse given by

$$(a,b)^{-1}=(a^{-1},-a^{-1}b). \tag{4}$$

Introduce coordinate functions f, h on the group manifold (the right half-plane), such that

$$\begin{aligned}\langle(a,b),f\rangle&=f\big((a,b)\big)=a,\\\langle(a,b),h\rangle&=h\big((a,b)\big)=b.\end{aligned}\tag{5}$$

What is the coproduct of f? A glance at (3) shows that $\Delta(f)=f\otimes f$. Indeed,

$$\begin{aligned}f\left((a,b)\cdot(a',b')\right)&\equiv\langle(a,b)\cdot(a',b'),f\rangle\\&=\langle(a,b)\otimes(a',b'),\Delta(f)\rangle\\&=\langle(a,b)\otimes(a',b'),f\otimes f\rangle\\&=\langle(a,b),f\rangle\langle(a',b'),f\rangle\\&=aa'\\&=f\left((aa',ab'+b)\right).\end{aligned}\tag{6}$$

Similarly, one finds $\Delta(h)=f\otimes h+h\otimes 1$, where the unit denotes the constant unit function on $A(1)$. The counits are

$$\epsilon(f)=f\big((1,0)\big)=1,\qquad\epsilon(h)=h\big((1,0)\big)=0. \tag{7}$$

Finally, from (4) one infers the antipode,

$$S(f)=\frac{1}{f},\qquad S(h)=-\frac{h}{f}. \tag{8}$$

3. Primitive Elements in the Hopf Algebra of Renormalization

3.1. The need for renormalization

Imagine playing an underwater bowling game. Any acceleration of the ball entails the acceleration of part of the water surrounding it, so that the mass m_{game} that you observe is the mass m_0 of the ball plus a correction Δm_0, due to

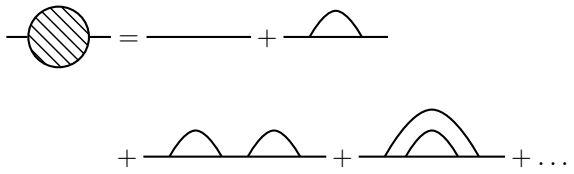
its interaction with the water. Or, consider a spherical conductor of mass m_0. A charge q is added to the conductor, which is now subjected to acceleration. The electrical self-interaction results in a net force opposing the acceleration, proportional (approximately) to the acceleration. Again, the observed mass will contain corrections to m_0 due to the interaction with the electromagnetic field.

Physicists studying these systems have the option of writing their dynamical equations in terms of the original, *bare* quantities (m_0), or the observed ones (m_{game}) — the passage between the two constitutes a *renormalization* of the theory. Using either set of quantities makes sense physically, because the interaction, in the above examples, can be turned off (playing bowling in the air, or discharging the conductor) so that either quantity is, at least in principle, measurable. Notice also that in both cases the correction Δm is finite, showing that renormalization is not intrinsically related to infinities.

Renormalization also appears in quantum field theory, but there it is a necessity, not an option. This is due to two reasons: first, the self-interactions of the quantum fields cannot be turned off, so that the bare quantities, corresponding to the free field case, are fictitious, measurable only in thought experiments. Second, the corrections to these fictitious quantities due to self-interactions are often infinite. Perturbative quantum field theory, nevertheless, treats the interactions as perturbations to the free field case. For example, in ϕ^3 theory, where the Lagrangian density is given by

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m_0^2\phi^2 - \frac{\lambda_0}{3!}\phi^3, \tag{9}$$

the last term is the self-interaction and its effects are calculated perturbatively. If λ_0 were zero, the observed mass of the corresponding particle would be m_0, the constant that appears explicitly in $\mathcal{L}$. But λ_0 is not zero, and the amplitude for propagation from event A to event B involves a sum over all possible trajectories compatible with the initial and final positions, including intermediate processes of emission and the subsequent reabsorption of virtual particles. Graphically this looks like



Each of the diagrams on the r.h.s. above stands for a particular, usually divergent, integral, which contributes to the mass correction. Infinities (in one-particle-irreducible diagrams) can be *primitive* (*e.g.* second term on the r.h.s. above) or *nested* (*e.g.* last term). The renormalization procedure consists of two steps: first *regularize, i.e.* find a (totally artificial) method to make all integrals converge, for example, by truncating the integration domain in momentum space down to some finite size. Second, rewrite the Lagrangian in terms of the observable parameters and remove the regularization. The expression of the observable parameters in terms of the bare ones will now involve (unobservable) infinities, but the functional dependence of the Lagrangian on the former will not. If this can be done consistently to all orders, the theory is *renormalizable*.

3.2. A toy model

Illustrating the above in the context of a realistic field theory involves rather hideous algebra. A good deal of the intricacies of the process though is captured by the following toy model [1]. There is a single primitive divergent integral $I(c)$, given by

$$I(c) = \int_0^\infty \frac{dy}{y+c}, \tag{10}$$

where c will be referred to as the *external parameter*. We represent this graphically by a single dot, $I(c) =$ •. Nested divergences are obtained by nesting $I(c)$. The simplest case is

$$I_2(c) = \int_0^\infty \frac{dy}{y+c}\int_0^\infty \frac{dz}{z+y}, \tag{11}$$

where the nesting is effected by letting the external parameter of the second appearance of $I(c)$ be the integration variable of the first — the corresponding graph is  , where the top dot refers to the first integral in (11) and the bottom one to the one nested inside it. When nesting twice, two possibilities exist

$$I_{3_1} = \int_0^\infty \frac{dy}{y+c}\int_0^\infty \frac{dz}{z+y}\int_0^\infty \frac{dw}{w+z}$$

$$I_{3_2} = \int_0^\infty \frac{dy}{y+c}\int_0^\infty \frac{dz}{z+y}\int_0^\infty \frac{dw}{w+y}, \tag{12}$$

with corresponding graphs  , respectively. Clearly, an infinite family of divergent integrals is produced, indexed by graphs known as *rooted trees* (*trees* because there are no closed circuits in them, *rooted* because the top vertex (= root) is special in that it has no parents) — these are the Feynman diagrams of our toy model.

We now apply a renormalization procedure to the above integrals. The regularization is done by modifying the measure, $dy \to y^{-\epsilon}dy$. The resulting convergent integrals can be expanded in a Laurent series in ϵ, the poles of which are recursively removed. For $I(c)$, one gets

$$I(c) \to I(\epsilon; c) = \int_0^\infty \frac{y^{-\epsilon}dy}{y+c}$$

$$= \frac{1}{\epsilon} - a + \mathcal{O}(\epsilon), \tag{13}$$

where $a \equiv \ln c$. The renormalized integral is obtained by subtracting from $I(\epsilon; c)$ its pole part and letting ϵ go to zero:

$$I^{\text{ren}}(c) = \lim_{\epsilon \to 0} \left(I(\epsilon; c) - \frac{1}{\epsilon} \right) = -a. \tag{14}$$

For the nested integrals, things are a little more complicated. Regularization proceeds as before, but the pole subtraction must be done recursively, starting with the innermost ("bottom", in the tree representation) divergences and working one's way up the tree towards the root. The renormalized value of the integral is given by a sum of 2^n terms, where n is the number of nested subdivergences, equal to the number of vertices of the corresponding tree — a very clear algorithmic description of the process can be found in Ref. 1.

As one can appreciate already from the above sketchy presentation, the renormalization process is far from unique. One may clearly regularize the integrals in an infinite number of ways and even the pole subtraction procedure can involve, for example, subtracting a finite part along with the poles. The resulting renormalized values of the integrals depend on these choices — it is one of the subtleties of the subject that the physics does not. To make this a little more plausible, consider an infinite homogeneous linear charge density and compute the electrostatic potential a radial distance $\rho = c$ away from the charge. Proceeding naïvely, one puts the zero of the potential at $\rho = \infty$ and integrates (minus) the electric field from infinity to c to find, in appropriate units

$$V(c) = -\int_{\infty}^{c} \frac{d\rho}{\rho} = \int_{0}^{\infty} \frac{dy}{y + c}, \tag{15}$$

our old friend! The problem is not in the physics, but in the choice of the reference point for the potential. To deal with it, one may choose a regularization $V(\epsilon; c)$ (as we did with $I(c)$ above), and then move the reference point to some finite distance ρ_0 by forming the difference $V(\epsilon; \rho) - V(\epsilon; \rho_0)$ and taking the limit $\epsilon \to 0$. In so doing, one is essentially subtracting the pole from $V(\epsilon; \rho)$, *plus an arbitrary finite amount which depends on* ρ_0. The physics, of course, lies in differences in the potential and, hence, is unaffected by changes in the subtraction procedure. A particular choice of regularization plus pole-removal procedure is referred to as a *renormalization scheme*. We summarize:

- there are infinitely many renormalization schemes to chose from
- the physics does not depend on the choice

3.3. Geometrization, part I

Consider the commutative algebra $\mathcal{H}$ generated by an infinite family of functions $\{\phi^T\}$, with T ranging over all rooted trees. A *character* on $\mathcal{H}$ is a linear map $\chi : \mathcal{H} \to \mathbb{R}$ that respects the algebra structure,

$$\chi(\phi^{T_1}\phi^{T_2}) = \chi(\phi^{T_1})\chi(\phi^{T_2}). \tag{16}$$

Call G the space of characters on $\mathcal{H}$. You can think of the ϕ's as functions on G, the value of $\phi^T \in \mathcal{H}$ on a particular point $\chi \in G$ being given by $\phi^T(\chi) \equiv \chi(\phi^T)$. Then the character property (16) corresponds to pointwise multiplication of the ϕ's, *i.e.*, $(\phi^{T_1}\phi^{T_2})(\chi) = \phi^{T_1}(\chi)\phi^{T_2}(\chi)$. Now, the renormalized value, in a certain renormalization scheme, of the product of two divergent integrals is defined to be the product of the renormalized values (in the same scheme) of the factors. This means that renormalization schemes are points in G. The value of ϕ^T on a point (scheme) g is the renormalized value of the integral corresponding to the tree T in the scheme corresponding to g. Furthermore, the ϕ's are coordinate functions on G, since, to completely specify a character, one needs only its values on the ϕ^T — the character property (plus linearity) then gives its values in the whole of $\mathcal{H}$. The beautiful discovery of Connes and Kreimer, who introduced the above setting [6], is that G is actually a (non-abelian) Lie group. Giving the group law consists in giving the coordinates $\phi^{T_3}(g_3)$ of the product $g_3 = g_1 g_2$, as smooth functions of the coordinates $\phi^{T_1}(g_1)$, $\phi^{T_2}(g_2)$ of the factors. First, two simple examples:

$$\phi^{\bullet}(g_1 g_2) = \phi^{\bullet}(g_1) + \phi^{\bullet}(g_2)$$

$$\phi^{\bullet\!\!\bullet}(g_1 g_2) = \phi^{\bullet\!\!\bullet}(g_1) + \phi^{\bullet\!\!\bullet}(g_2) + \phi^{\bullet}(g_1)\phi^{\bullet}(g_2);$$

then, a not-so-simple example:

$$\begin{aligned}\phi^{\bullet\!\diagup\!\diagdown\!\bullet}(g_1 g_2) = {} & \phi^{\bullet\!\diagup\!\diagdown\!\bullet}(g_1) + \phi^{\bullet\!\diagup\!\diagdown\!\bullet}(g_2) \\ & + 2\phi^{\bullet}(g_1)\phi^{\bullet\!\!\bullet}(g_2) \\ & + \left(\phi^{\bullet}(g_1)\right)^2 \phi^{\bullet}(g_2);\end{aligned}$$

finally, the general formula:

$$\begin{aligned}\phi^T(g_1 g_2) = {} & \phi^T(g_1) + \phi^T(g_2) \\ & + \sum_{\text{cuts } C} \phi^{P^C(T)}(g_1)\phi^{R^C(T)}(g_2).\end{aligned} \tag{17}$$

The sum is over all *admissible cuts* C of the tree T. An admissible cut can be either a *simple cut* or a *composite cut*. The former involves cutting an edge of the tree and discarding the half-edges produced (but *not* the vertices to which these are attached). In this way, the tree is separated into two subtrees, one that contains the root (denoted by $R^C(T)$ above), and one that "falls to the floor"[i] after the cut (denoted by $P^C(T)$ above). A composite cut, on the other hand, involves cutting $k > 1$ edges, with the constraint that there is at most one cut on any path from the root downwards. In this case, $R^C(T)$ again denotes the subtree containing the root, but $P^C(T)$ now denotes the collection $\{T_i\}$, $1 \leq i \leq k$ of subtrees that fall to the floor, with $\phi^{P^C(T)}$ denoting the product $\phi^{T_1}\phi^{T_2}\ldots\phi^{T_k}$. It can be shown that the above group law is associative. It is easy to read off the coproduct of ϕ^T from (17):

$$\begin{aligned}\Delta(\phi^T) = {} & \phi^T \otimes 1 + 1 \otimes \phi^T \\ & + \sum_{\text{cuts } C} \phi^{P^C(T)} \otimes \phi^{R^C(T)}.\end{aligned} \tag{18}$$

The units appearing on the r.h.s. denote the constant unit function on G. From this expression one finds a recursive formula for the antipode,

$$S(\phi^T) = -\phi^T - \sum_{\text{cuts } C} S(\phi^{P^C(T)})\phi^{R^C(T)}. \qquad (19)$$

The counit of all ϕ's is zero, while that of the unit function is, of course, one. Given then any two characters (in particular, two renormalization schemes), one may form their product according to the above formulas to obtain a new one. The counit just given implies that the unit character, wrt this product, assigns the value zero to all divergent integrals, while assigning 1 to the unit function. This completes the Hopf algebra structure of $\mathcal{H}$, dual to the Lie group structure of G.

But, apart from aesthetics, what is it good for? The answer lies, partly, in the following observation [1]. Define a map R that encodes the subtraction procedure, *e.g.* by assigning to each ϕ^T the pole part of its Laurent expansion, evaluated at $c = 1^{ii}$. Use R to define a *twisted antipode* S_R via

$$S_R(\phi^T) = -R(\phi^T) - R\left(\sum_{\text{cuts } C} S_R(\phi^{P^C(T)})\phi^{R^C(T)}\right) \qquad (20)$$

(notice the similarity of the recursive structure with that of (19)). Then all of renormalization's recursive complexity is encoded neatly in the formula

$$\phi^T_{\text{ren}} = S_R(\phi^T_{(1)})\phi^T_{(2)}\Big|_{\epsilon=0}. \qquad (21)$$

Example 1 *Renormalization of* $\phi^{\bullet\bullet}$

From the coproduct

$$\Delta(\phi^{\bullet\bullet}) = \phi^{\bullet\bullet} \otimes 1 + 1 \otimes \phi^{\bullet\bullet} + \phi^{\bullet} \otimes \phi^{\bullet} \qquad (22)$$

and Eq. (21), one gets

$$\phi^{\bullet\bullet}_{\text{ren}} = S_R(\phi^{\bullet\bullet}) + \phi^{\bullet\bullet} + S_R(\phi^{\bullet})\phi^{\bullet}, \qquad (23)$$

the limit $\epsilon \to 0$ being implied. For the twisted antipodes, (20) gives

$$\begin{aligned} S_R(\phi^{\bullet}) &= -R(\phi^{\bullet}) \\ S_R(\phi^{\bullet\bullet}) &= -R(\phi^{\bullet\bullet}) - R(-R(\phi^{\bullet})\phi^{\bullet}). \end{aligned} \qquad (24)$$

Substituting these in (23) gives $\phi^{\bullet\bullet}_{\text{ren}}$ as a sum of 4 ($= 2^2$) terms,

$$\begin{aligned} \phi^{\bullet\bullet}_{\text{ren}} = \phi^{\bullet\bullet} - R(\phi^{\bullet\bullet}) - R(\phi^{\bullet})\phi^{\bullet} \\ + R(R(\phi^{\bullet})\phi^{\bullet}). \end{aligned} \qquad (25)$$

Evaluating the regularized integrals and expanding in Laurent series in ϵ, one finds (with some abuse of notation)

$$\phi^{\bullet} = \frac{1}{\epsilon} - a + \left(\frac{a^2}{2} + \frac{\pi^2}{6}\right)\epsilon + \mathcal{O}(\epsilon^2) \qquad (26)$$

$$\phi^{\bullet\bullet} = \frac{1}{2\epsilon^2} - \frac{a}{\epsilon} + a^2 + \frac{5\pi^2}{12} + \mathcal{O}(\epsilon), \qquad (27)$$

so that, finally,

$$\phi^{\bullet\bullet}_{\text{ren}} = \frac{a^2}{2} + \frac{\pi^2}{4}. \qquad (28)$$

3.4. Primitive elements in $\mathcal{H}$

As the last example probably makes clear, renormalizing multiply nested integrals can be heavy work. A glance at (21) shows that the complexity of the task, for a particular ϕ^T, depends on the number of terms in the coproduct of ϕ^T. In fact, one realizes that this dependence is even stronger by taking into account (20), *i.e.* the fact that the complexity of the twisted antipodes produced by (21) depends itself on that of the coproduct. It would be nice then if the coproduct of the ϕ's were simpler. But it isn't. The next best thing is to look for a new set of coordinates on G with simpler coproducts. For example, one cannot ask for anything more from $\phi^{\bullet}$:

$$\Delta(\phi^{\bullet}) = \phi^{\bullet} \otimes 1 + 1 \otimes \phi^{\bullet}, \qquad (29)$$

(such a coproduct will be called *1-primitive*) but $\phi^{\bullet\bullet}$ isn't as innocent, see (22) (and Ex. 1 for the consequences). A little experimentation though shows that

$$\psi^{\bullet\bullet} = \phi^{\bullet\bullet} - \frac{1}{2}(\phi^{\bullet})^2 \qquad (30)$$

also has a 1-primitive coproduct (notice that the coproduct of a product is the product of the coproducts). To renormalize $\psi^{\bullet\bullet}$ then, two options are available. The first one is to write

$$\psi^{\bullet\bullet}_{\text{ren}} = \phi^{\bullet\bullet}_{\text{ren}} - \frac{1}{2}(\phi^{\bullet}_{\text{ren}})^2, \qquad (31)$$

and renormalize the ϕ's on the r.h.s.. Since

$$\phi^{\bullet}_{\text{ren}} = \phi^{\bullet} - R(\phi^{\bullet})\Big|_{\epsilon=0} = -a, \qquad (32)$$

one finds, using the result of Ex. 1, Eq. [28], $\psi^{\bullet\bullet}_{\text{ren}} = \pi^4/4$. The second option is to use directly (21), which holds for functions of the coordinates as well, since both the coproduct and the twisted antipode are algebra homomorphisms — this gives $\psi^{\bullet\bullet}_{\text{ren}} = \psi^{\bullet\bullet} - R(\psi^{\bullet\bullet})$. But a Laurent expansion of $\psi^{\bullet\bullet}$ shows that

$$\psi^{\bullet\bullet} = \frac{\pi^2}{4} + \mathcal{O}(\epsilon), \qquad (33)$$

i.e. $\psi^{\bullet\bullet}$ does not even have poles, so that $R(\psi^{\bullet\bullet}) = 0$, and one immediately recovers the above result. Clearly, there are good reasons to further investigate the applicability of the second option. Consider, for example, renormalizing a ϕ with ten vertices. The analogue of (25) would then contain $2^{10} = 1024$ terms. On the other hand, imagine that appropriate additions to that ϕ turn it into a 1-primitive ψ. Then only two terms would appear in its renormalization. Too good to be true? Read on. Some pertinent questions then are: Can one find other ψ's with 1-primitive coproduct? Can all ϕ's be

traded for 1-primitive ψ's? The answer is yes and no, in that order. A little more experimentation, for example, shows that

$$\psi^{\bullet\!\!\bullet\!\!\bullet} = \phi^{\bullet\!\!\bullet\!\!\bullet} - \phi^{\bullet}\phi^{\bullet\!\!\bullet} + \frac{1}{3}(\phi^{\bullet})^3 \tag{34}$$

is also 1-primitive. On the other hand, if a set of 1-primitive coordinates could be chosen on G, then G would be abelian, which it is not. The most one can hope for then is to make an optimal choice of coordinates that best adapts to the "abelian directions" on G.

3.5. Geometrization, part II

"Ladder" (linear) trees provide a convenient point of entry to the problem of primitive elements in $\mathcal{H}$. It is easy to see that their coproduct is given by

$$\Delta(\phi^{(n)}) = \sum_{k=0}^{n} \phi^{(k)} \otimes \phi^{(n-k)}\,, \tag{35}$$

where $\phi^{(r)}$ denotes the ladder tree with r vertices ($\phi^{(0)}$ stands for the unit function). But this form of the coproduct rings a bell. Consider the commutative algebra of power series of the form $g = 1 + c_1 x + c_2 x^2 + \ldots$, with the usual product. Introducing coordinates $\{\xi_n\}$ on the space of such power series, with $\langle g, \xi_n\rangle = c_n$, one easily finds that the dual to the product of power series is the coproduct

$$\Delta(\xi_n) = \sum_{k=0}^{n} \xi_k \otimes \xi_{n-k} \tag{36}$$

(compare with (35)). But series of the above form can also be written as exponentials,

$$g = e^{c_1' x + c_2' x^2 + \cdots}. \tag{37}$$

Change now the coordinates from ξ to ξ', such that $\langle g, \xi_n'\rangle = c_n'$. Under multiplication of series, the constants c' simply add up, implying the coproduct $\Delta(\xi_n') = \xi_n' \otimes 1 + 1 \otimes \xi_n'$, *i.e.* the ξ_n' are all 1-primitive. The obvious isomorphism with the Hopf algebra of ladder trees implies that the $\psi^{(n)}$ defined by

$$e^{\sum_{n=1}^{\infty} \psi^{(n)} x^n} = \sum_{r=0}^{\infty} \phi^{(r)} x^r \tag{38}$$

are all primitive. (38) may be inverted to give

$$\psi^{(n)} = \frac{1}{n!}\frac{\partial^n}{\partial x^n} \log\left(\sum_{m=0}^{\infty} \phi^{(m)} x^m\right)\Bigg|_{x=0}, \tag{39}$$

putting the problem of the ladder 1-primitives, including the ten-vertex one, to rest.

This may look, so far, like a happy combinatorial accident. But it actually points to the deeper geometrical origin of 1-primitiveness. There is no space here to delve into the details, but a sketch of some of the ideas will be given (for details, see [4]). First, renormalization schemes are written as exponentials of elements of $\mathfrak{g}$, the Lie algebra of G:

$$g = e^{\sum_T c^T Z_T}\,, \tag{40}$$

where Z_T are the generators of $\mathfrak{g}$. Then new coordinates ψ^T are introduced, such that

$$\langle g, \psi^T\rangle = c^T. \tag{41}$$

These are, of course, the *normal coordinates* on G. Their coproduct is computed from the Baker-Cambell-Hausdorff (BCH) formula,

$$\Delta(\psi^A) = \psi^A \otimes 1 + 1 \otimes \psi^A + \frac{1}{2} f_{B_1 B_2}{}^A \psi^{B_1} \otimes \psi^{B_2} + \ldots, \tag{42}$$

where $f_{B_1 B_2}{}^A$ are the structure constants of $\mathfrak{g}$. Some of the generators of $\mathfrak{g}$ cannot be written as commutators. For the dual ψ's, the third term above, and all higher ones, are zero, *i.e.*, they are 1-primitive. This observation points to a natural generalization of the concept of 1-primitiveness. There are, for example, some other generators of $\mathfrak{g}$ that *can* be written as commutators, but not as double commutators. The dual ψ's will have the third term on the r.h.s. of (42) present, but all higher terms equal to zero — they are to be called, accordingly, 2-primitive. What enters naturally then in to the discussion of k-primitiveness is the lower central series of $\mathfrak{g}$, which classifies its generators in classes $\mathfrak{g}_k$, according to the maximal number k of nested commutators by which they can be produced.

But there is more to be gained from our geometric approach. For any function f on G, one may extend the exterior differential of f at the identity, $df|_e$, to a left-invariant 1-form Π_f on G. If f is quadratic or of a higher degree in the ϕ's, $df|_e$, and hence Π_f, vanishes. Given that the linear part of ψ^T is ϕ^T, one concludes that $\Pi_{\psi^T} = \Pi_{\phi^T}$. For a primitive ψ^T, on the other hand, the general formula

$$\Pi_f = S(f_{(1)}) df_{(2)} \tag{43}$$

shows that $\Pi_{\psi^T} = d\psi^T$, and hence, $\Pi_{\phi^T} = d\psi^T$ is closed. Application of the inverse Poincaré lemma to $\Pi_{\phi^T} = S(\phi^T_{(1)}) d\phi^T_{(2)}$ then provides an expression of the 1-primitive ψ^T as the co-cone (potential) of the closed Π_{ϕ^T}. A little further trickery, which takes into account the fact that the coproduct of the ϕ's is linear in the ϕ's in the right tensor factor, results in an elegant formula,

$$\psi^T = -\frac{1}{\Phi} S(\phi^T), \tag{44}$$

where Φ counts the monomial order of the ϕ's, $\Phi(\phi^{T_1} \ldots \phi^{T_k}) = k\, \phi^{T_1} \ldots \phi^{T_k}$.

Some *very* rough estimate of the savings in, say, CPU time, from switching to the normal coordinates can be obtained by assigning a cost of 2^k to a k-primitive element, regardless of the number of vertices of its index, while assuming that a ϕ with n vertices costs 2^n. The numbers $P_{n,k}$

of k-primitive elements with $n > k$ vertices are given by the generating function [2]

$$P_k(x) \equiv \sum_{n=1}^{\infty} P_{n,k} x^n = \sum_{s|k} \frac{\mu(s)}{k} \Big(1 - \prod_{n=1}^{\infty} \big(1 - x^{ns}\big)^{r_n}\Big)^{k/s}. \quad (45)$$

The sum on the r.h.s. above extends over all divisors s of k, including 1 and k. $\mu(s)$ is the Möbius function, equal to zero, if s is divisible by a square, and to $(-1)^p$, if s is the product of p distinct primes ($\mu(1) \equiv 1$). The asymptotic behavior of $P_{n,k}$, for large values of n, is [2]

$$f_k \equiv \lim_{n\to\infty} \frac{P_{n,k}}{r_n} = \frac{1}{c}\left(1 - \frac{1}{c}\right)^{k-1}, \quad (46)$$

where $c = 2.95\ldots$ is the Otter constant and r_n is the number of rooted trees with n vertices. This is good news, as the population of the CPU-intensive high-k ψ's is seen to be exponentially suppressed. The ratio of the total costs of renormalizing all generators with n vertices in the two bases then is

$$\rho_n = \frac{r_n 2^n}{\sum_{k=1}^{n-1} P_{n,k} 2^k} \approx (c-2)\Big(\frac{c}{c-1}\Big)^{n-1}, \quad (47)$$

which soars to 5.4×10^5, a year's worth of minutes, for $n = 33$.

More insights, from a geometrical point of view, can be found in [4]. Very significant progress has been made also by mostly algebraic means, see [9] and, more recently, [7, 8].

4. Primitive Elements in the Hopf Algebra of Quantum Measures

4.1. Quantum mechanics as a quantum measure theory

A second example where geometry, in the guise of Hopf algebras, illuminates problems of an algebraic/combinatoric nature is furnished by Sorkin's proposal of a generalization of quantum mechanics [12–14] — Ref. [3] is followed closely in the sequel. Consider the standard two-slit interference experiment and call H the set of all electron histories (worldlines) leaving the electron gun and arriving at the detector at specified time instants (to avoid technicalities, consider H to be measurable). Denote by A (B) the subset of H consisting of all histories in which the electron passes through slit a (b), ignoring the possibility of the electron winding around both slits. There are four possible ways of blocking the two slits — denote by P_{ab}, P_a, P_b and $P_0 = 0$ the corresponding probabilities of arrival at the detector, the last one corresponding to both slits being blocked off. Sorkin's approach is to consider these probabilities as the values of a certain measure function μ defined on the set of subsets of H, *e.g.* $P_a = \mu(A)$. When mutually exclusive alternatives exist, as when both slits are open, the union of the corresponding (disjoint) subsets is to be taken, *e.g.* $P_{ab} = \mu(A \sqcup B)$ ($\sqcup$ denotes disjoint union). Physical theories are distinguished by the measures they employ, for example, classical mechanics uses a "linear" measure $\mu_{\rm cl}$, satisfying the sum rule

$$I_2^{\mu_{\rm cl}}(A,B) \equiv \mu_{\rm cl}(A \sqcup B) - \mu_{\rm cl}(A) - \mu_{\rm cl}(B) = 0, \quad (48)$$

and hence fails to predict any interference. Quantum mechanics, on the other hand, uses $\mu_{\rm q}$, satisfying $I_2^{\mu_{\rm q}}(A,B) \neq 0$. The interesting observation by Sorkin is that in a three slit experiment (with eight possibilities for blocking the slits), the probabilities predicted by quantum mechanics *do* satisfy the sum rule

$$\begin{aligned} I_3^{\mu_{\rm q}}(A,B,C) \equiv\ & \mu_{\rm q}(A \sqcup B \sqcup C) \\ & - \mu_{\rm q}(A \sqcup B) - \mu_{\rm q}(A \sqcup C) - \mu_{\rm q}(B \sqcup C) \\ & + \mu_{\rm q}(A) + \mu_{\rm q}(B) + \mu_{\rm q}(C) \\ & = 0. \end{aligned} \quad (49)$$

It is easy to show that $\mu_{\rm cl}$ also satisfies (49), as a result of (48). There is an obvious generalization to the k-slit experiment, involving the symmetric functional I_k^{μ}, given by

$$\begin{aligned} I_k^{\mu}(A_1,\ldots,A_k) \equiv\ & \mu(A_1 \sqcup \ldots \sqcup A_k) \\ & - \sum_i \mu(A_1 \sqcup \ldots \sqcup \hat{A}_i \sqcup \ldots \sqcup A_k) \\ & + \sum_{i<j} \mu(A_1 \sqcup \ldots \sqcup \hat{A}_i \sqcup \ldots \\ & \ldots \sqcup \hat{A}_j \sqcup \ldots \sqcup A_k) \\ & \ldots \\ & + (-1)^{k+1} \sum_i \mu(A_i), \end{aligned} \quad (50)$$

where symbols with hats are omitted and all A_i are mutually disjoint. Due to the recursion relation

$$\begin{aligned} I_{k+1}^{\mu}(A_0,A_1,\ldots,A_k) = & I_k^{\mu}(A_0 \sqcup A_1, A_2, \ldots, A_k) \\ & - I_k^{\mu}(A_0, A_2, \ldots, A_k) \\ & - I_k^{\mu}(A_1, A_2, \ldots, A_k), \end{aligned} \quad (51)$$

the sum rule $I_{k+1}^{\mu} = 0$ follows from $I_k^{\mu} = 0$. It is natural now to contemplate a family of theories, indexed by a positive integer k, defined by the sum rule $I_{k+1}^{\mu} = 0$, with $I_k^{\mu} \neq 0$ for the corresponding measure. In this scheme, classical mechanics is a $k = 1$ theory while quantum mechanics corresponds to $k = 2$.

The above formulas for I_k^{μ} need to be extended to the general case, *i.e.* when the arguments are possibly overlapping sets. For the $k = 2$ case, Sorkin shows that bilinearity

implies the following equivalent forms

$$\begin{aligned} I_2^\mu &= \mu(A\cup B)+\mu(A\cap B)-\mu(A\backslash B)-\mu(B\backslash A)\\ &= \mu(A\triangle B)+\mu(A)+\mu(B)\\ &\quad -2\mu(A\backslash B)-2\mu(B\backslash A). \end{aligned} \tag{52}$$

The symbol $\backslash$ above denotes a set-theoretic difference while $\triangle$ denotes a symmetric difference.

4.2. Coderivatives

A brief excursion further into Hopf algebraic territory is necessary at this point. One way of looking at the coproduct of a function is as an indefinite translation. The right translation R_g on the group is defined by $R_g(g')=g'g$. Its pullback on functions $R_g^*(f)\equiv f_g$ is given by

$$f_g(g')=f(g'g)=f_{(1)}(g')f_{(2)}(g).$$

One infers that $f_{(1)}(\cdot')f_{(2)}(g)$ is the right-translate of f by g, while $f_{(1)}(\cdot')f_{(2)}(\cdot)$, a function of two arguments, is the indefinitely translated f: its second argument defines the translation while the first evaluates the translated function (left translations can be similarly handled exchanging the two tensor factors of the coproduct). Introduce now the operator $\mathcal{L}:\mathcal{A}\mapsto\mathcal{A}\otimes\mathcal{A}$, defined by

$$\mathcal{L}f=\Delta(f)-f\otimes 1. \tag{53}$$

The above way of interpreting the coproduct shows that $\mathcal{L}$ can be considered an indefinite discrete derivative or *coderivative* for short:

$$\begin{aligned} (\mathcal{L}f)(g',\,g) &= \langle f_{(1)}\otimes f_{(2)}-f\otimes 1, g'\otimes g\rangle\\ &= f(g'g)-f(g'). \end{aligned} \tag{54}$$

When g is close to the identity, $g=e+X+\ldots$, with X in the Lie algebra of the group, $(\mathcal{L}f)(\cdot',g)$ is approximately (proportional to) the derivative of f along the left invariant vector field corresponding to X. Higher order coderivatives $\mathcal{L}^k f$ can similarly be defined, with the understanding that the successive applications of $\mathcal{L}$ are to be taken at the leftmost tensor factor,

$$\mathcal{L}^k f\equiv(\mathcal{L}\otimes\mathrm{id})\circ\mathcal{L}^{k-1}f\,,\qquad k=2,3,\ldots, \tag{55}$$

so that, for example,

$$\begin{aligned} \mathcal{L}^2 f &\equiv (\mathcal{L}\otimes\mathrm{id})\circ\mathcal{L}f\\ &= (\mathcal{L}\otimes\mathrm{id})(f_{(1)}\otimes f_{(2)}-f\otimes 1)\\ &= f_{(1)}\otimes f_{(2)}\otimes f_{(3)}-f_{(1)}\otimes 1\otimes f_{(2)}\\ &\quad -f_{(1)}\otimes f_{(2)}\otimes 1+f\otimes 1\otimes 1. \end{aligned} \tag{56}$$

Of particular interest is the evaluation of a k-th order coderivative at the identity of the group, $(\mathcal{L}^k f)(e,\cdot,\ldots)\equiv(\mathcal{L}^k f)(e)$, *e.g.*,

$$\begin{aligned} (\mathcal{L}f)(e) &= f-\epsilon(f)1\\ (\mathcal{L}^2 f)(e) &= f_{(1)}\otimes f_{(2)}-f\otimes 1-1\otimes f\\ &\quad +\epsilon(f)1\otimes 1. \end{aligned} \tag{57}$$

It is now possible to introduce formally the notion of k-*primitiveness*: a function f will be called k-*primitive* if all its coderivatives of order r at the identity, $(\mathcal{L}^r f)(e)$, with $r>k$, are equal to zero, while $(\mathcal{L}^k f)(e)$ is not.

4.3. Quantum measures and k-primitiveness

4.3.1. The abelian group of histories

Referring back to the k-slit experiment, call H the set of histories available to a particle, taken as a measurable set for simplicity. One may deal with a given subset A of H in terms of its *characteristic function*, defined by $\chi_A(x)=1$ if $x\in A$, $\chi_A(x)=0$ if $x\in H\backslash A$. Denote by G the set of all linear combinations of characteristic functions of measurable subsets of H[iii]. A typical element g of G is of the form $g=\lambda_1\chi_{A_1}+\lambda_2\chi_{A_2}+\ldots$, where the A_i are measurable subsets of H and $\lambda_i\in\mathbb{C}$. G may be turned into an abelian group with the group law given by addition. Then the identity e is the zero function, $e=\chi_\emptyset=0$, and the inverse of g is $-g$.

As in Sorkin's approach, a physical theory derives its probabilities from a measure function μ, defined now on G, *e.g.* $P_a=\mu(\chi_A)$ in the two-slit experiment. In the presence of mutually exclusive alternatives, the *sum of the characteristic functions of the corresponding subsets* is to be taken — this corresponds to the disjoint union in terms of the subsets themselves, as in Refs. 11 and 12. What is attractive though in working with characteristic functions is that, by extending this definition (*i.e.*, addition of the characteristic functions) to non-disjoint subsets, we recover the rather complicated interference term (52) and its generalizations. Consider for example a quadratic functional μ^2, with μ additive, evaluated on two overlaping subsets A and B — the resulting interference term is

$$\begin{aligned} I_2^{\mu^2} &= \mu(\chi_A+\chi_B)^2-\mu(\chi_A)^2-\mu(\chi_B)^2\\ &= 2\mu(\chi_A)\mu(\chi_B)\\ &= 2\big(\mu(\chi_{A\backslash B})\mu(\chi_{B\backslash A})+\mu(\chi_{A\backslash B})\mu(\chi_{A\cap B})\\ &\quad +\mu(\chi_{A\cap B})\mu(\chi_{B\backslash A})+\mu(\chi_{A\cap B})^2\big), \end{aligned} \tag{58}$$

where, in the last step, the substitution $\chi_A=\chi_{A\backslash B}+\chi_{A\cap B}$ was used, and similarly for χ_B. On the other hand, the first, for example, of (52) becomes

$$\begin{aligned} I_2^{\mu^2} &= \mu(\chi_{A\cup B})^2+\mu(\chi_{A\cap B})^2-\mu(\chi_{A\backslash B})^2\\ &\quad -\mu(\chi_{B\backslash A})^2. \end{aligned} \tag{59}$$

Substituting $\chi_{A\cup B}=\chi_{A\backslash B}+\chi_{B\backslash A}+\chi_{A\cap B}$ and expanding, one recovers the right hand side of (58).

The quantum measures considered up to now are elements of the commutative Hopf algebra $\mathcal{A} \equiv C^\infty(G)$ of smooth functions on G. Their counit vanishes, since $\mu(\emptyset) = 0$. The linearity of the classical measure, Eq. (48), implies that $\mu_{\rm cl}(\chi_A + \chi_B) = \mu_{\rm cl}(\chi_A) + \mu_{\rm cl}(\chi_B)$, or, in terms of the coderivative,

$$0 = (\mathcal{L}^2\mu)(e) = \mu_{\rm cl\,(1)} \otimes \mu_{\rm cl\,(2)} - \mu_{\rm cl} \otimes 1 \\ -1 \otimes \mu_{\rm cl} + \epsilon(\mu_{\rm cl}) 1 \otimes 1, \qquad (60)$$

the last term being zero. Hence, $\mu_{\rm cl}$ is a 1-primitive element of $\mathcal{A}$. More generally, the following lemma holds:

Lemma 1 The symmetric functionals I^μ_k, defined in Eq. (50), coincide with the k-th order coderivatives $(\mathcal{L}^k\mu)(e)$ of Eq. (55).

The straightforward inductive proof is left as an exercise. The main result of this section may now be stated:

Proposition 1 In the algebra $\mathcal{A}$ of functions on G, every k-primitive element is a k-th degree polynomial in 1-primitive elements.

The proof can be given in a number of ways. For example, it is easily established that $\mathcal{A}$ is a cocommutative graded connected Hopf algebra. This means that the coproduct is symmetric under exchange of the two tensor factors, that there is a grading respected by the coproduct (k-primitiveness), and that the only elements with zero grade are numbers. Then, application of the Milnor-Moore theorem [10] gives that $\mathcal{A}$ is isomorphic to the universal enveloping algebra of its subalgebra of 1-primitive elements, which establishes the proposition. Alternatively, and closer to the spirit of this paper, one may introduce normal coordinates on H. Because H is abelian the normal coordinates are 1-primitive. Then the vanishing at the identity of all coderivatives, of order higher than k (but not that of order k), of a function of the coordinates implies that the function is a polynomial of order k in the coordinates, q.e.d.

5. Conclusions

It has been argued that a geometric setting, cast in Hopf algebraic language, might be efficient in dealing with problems of a combinatoric nature. Two examples were used to illustrate the point:

- A toy-model renormalization is simplified significantly by introducing normal coordinates on the infinite dimensional Lie group of renormalization schemes of Connes and Kreimer, while 1-primitive functions on the group are shown to correspond to closed left-invariant 1-forms.
- Quantum measures of k-th order, proposed by Sorkin as generalizations of quantum mechanics, are described as k-primitive functions on the group of histories associated whit a particular experiment. As a result, they are shown to be polynomials of order k in additive (*i.e.*, classical-like) measures.

It would be nice to see this list grow longer fast.

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i. A mental picture borrowed from D. Kreimer.

ii. In general, R must satisfy

$$R(xy) - R(R(x)y) - R(xR(y)) + R(x)R(y) = 0$$

— this guarantees that S_R (see below) is multiplicative.

iii. These are known as *simple* functions on H.

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