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Classical and Quantum $sl(1|2)$ Superalgebras, Casimir Operators and Quantum Chain Hamiltonians

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Abstract

We examine in this paper the two parameter deformed superalgebra $\mathcal{U}_{qs}(sl(1|2))$ and use the results in the construction of quantum chain Hamiltonians. This study is done both in the framework of the Serre presentation and in the R-matrix scheme of Faddeev, Reshetikhin and Takhtajan (FRT). We show that there exists an infinite number of Casimir operators, indexed by integers $p \geq 2$ in the undeformed case and by $p \in \mathbb{Z}$ in the deformed case, which obey quadratic relations. The construction of the dual superalgebra of functions on $SL_{qs}(1|2)$ is also given and higher tensor product representations are discussed. Finally, we construct quantum chain Hamiltonians based on the Casimir operators. In the deformed case we find two Hamiltonians which describe deformed $t-J$ models.

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1 Introduction

A Lie algebra $\mathcal{G}$ being given, it is known that the coproduct of the elements of the centre $C(\mathcal{G})$ of $\mathcal{G}$, evaluated in a representation of $\mathcal{G}$, gives two-site quantum chain Hamiltonians, which are invariant under the Lie algebra $\mathcal{G}$ under consideration. A physically important case is the $t-J$ model at the supersymmetric point which can be obtained by the above procedure using the second order Casimir operator and the fundamental representation of the simple Lie superalgebra $sl(1|2)$. Recently, the authors of ref. [4] found a q-deformation of the $t-J$ model, invariant under the quantum group $U_q(sl(1|2))$.

However, there are reasons to wonder whether the Hamiltonians one can construct in this way are unique.

A simple Lie superalgebra being given, there exist many inequivalent simple root bases (*i.e.* many Cartan matrices) [6] which can be used to describe the superalgebra in the Serre presentation [14]. Unlike in the classical case, where the Hopf structure is essentially unique, the quantum deformation of $\mathcal{U}(sl(1|2))$ exhibits the novel feature that the Hopf structures that are natural in each basis are not related by the transformation law between the corresponding bases. One can thus suspect that they lead to unequal Hamiltonians.

On the other hand, different Casimir operators may also lead, through the above procedure, to different Hamiltonians.

It is our purpose, in this paper, to explore these possibilities. Accomplishing this necessitates an efficient formalism for dealing with quantum superalgebras. There are already in the literature approaches to an FRT-type [3] formulation of quantum superalgebras [1, 10] which, however, deal only with universal enveloping superalgebras (see also [8, 11, 15] for alternative approaches from a different perspective). As our study calls for representations of elements of the above algebra (or its tensor powers), we give a new FRT construction of $\mathcal{U}_{qs}(sl(1|2))$, complete with its dual superalgebra of functions on $SL_{qs}(1|2)$.

The paper is organised as follows. In Sect. 2 we recall the structure of the classical superalgebra $sl(1|2)$, using the Serre presentation in the fermionic basis. We show that the centre of its universal enveloping superalgebra $\mathcal{U}(sl(1|2))$ contains an infinite set of Casimir operators, indexed by integers $p \geq 2$, that satisfy quadratic relations. The quantum analogue $\mathcal{U}_{qs}(sl(1|2))$ of $\mathcal{U}(sl(1|2))$, with two deformation parameters, is considered in Sect. 3. The corresponding deformed Casimir operators, indexed by $p \in \mathbb{Z}$ are given. They also obey quadratic relations. In Sect. 4 we recast $\mathcal{U}_{qs}(sl(1|2))$ in its FRT form, introducing a suitable R-matrix and matrices of generators $\underline{L}^{\pm}$. We also discuss available choices in the construction of the dual Hopf algebra. A bosonised basis for the algebra is given in Sect. 5 and the representation of the coproduct of an infinite set of Casimirs is computed. Finally, in Sect. 6, the above results are used to construct three-state (per site) quantum chain Hamiltonians. The $sl(1|2)$-invariant Hamiltonian is unique (up to the identity and a normalization factor) and describes the well-known $t-J$ model at the supersymmetric point. A similar construction in the case of $\mathcal{U}_{qs}(sl(1|2))$-invariant Hamiltonians provides two

qs-deformations of the supersymmetric $t-J$ model. In the case of open boundary conditions, an equivalence transformation eliminates all but the usual deformation parameter q, leading to two Hamiltonians, one being the $(1,2)$ Perk–Schultz Hamiltonian [12]. The results are finally summarised in Sect. 7.

2 The classical superalgebra $\mathcal{U}(sl(1|2))$

Before studying the quantum case, we would like to recall some features and do some comments about the (undeformed) simple Lie superalgebra $sl(1|2)$. A classical simple Lie superalgebra can be described in many inequivalent bases (*i.e.* there exist many inequivalent simple root systems), each one being associated to a particular Dynkin diagram. The different possible Dynkin diagrams can be found by applying generalised Weyl transformations associated to the fermionic roots of the diagrams, until no new diagram appears. In the case of $sl(1|2)$, the two possibilities are the so-called distinguished and fermionic bases.

2.1 Presentation in the fermionic basis

Denoting by H_i $(i=1,2)$ and $E_i^{\pm}$ $(i=1,2,3)$ the generators of $sl(1|2)$, the $\mathbb{Z}_2$-gradation in the fermionic basis is such that H_1, H_2 and $E_3^{\pm}$ are even elements and $E_1^{\pm}$, $E_2^{\pm}$ are odd elements, that is

$$\deg H_1 = \deg H_2 = \deg E_3^{\pm} = 0 \quad \text{and} \quad \deg E_1^{\pm} = \deg E_2^{\pm} = 1 \tag{2.1}$$

where $\deg X$ stands for the degree of the generator X.

In the Serre presentation, the commutation relations in the fermionic basis read

$$\begin{aligned}
&[H_i, H_j] = 0 \quad \text{and} \quad [H_i, E_j^{\pm}] = \pm a_{ij} E_j^{\pm} \, , \\
&\{E_1^{\pm}, E_1^{\pm}\} = \{E_2^{\pm}, E_2^{\pm}\} = \{E_1^{\pm}, E_2^{\mp}\} = 0 \, , \\
&\{E_1^{+}, E_1^{-}\} = H_1 \, , \\
&\{E_2^{+}, E_2^{-}\} = H_2 \, ;
\end{aligned} \tag{2.2}$$

(a_{ij}) is the Cartan matrix in the fermionic basis:

$$(a_{ij}) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} . \tag{2.3}$$

The generators $E_3^{\pm}$ are defined by

$$E_3^{\pm} = \{E_1^{\pm}, E_2^{\pm}\} \tag{2.4}$$

and the Serre relations can be written as

$$[E_1^{\pm}, E_3^{\pm}] = [E_2^{\pm}, E_3^{\pm}] = 0 \, . \tag{2.5}$$

It follows from the previous equations that the remaining relations among the generators read

$$\begin{aligned} &[E_3^+, E_1^-] = E_2^+ \;, \qquad && [E_3^+, E_2^-] = E_1^+ \;, \\ &[E_3^-, E_1^+] = -E_2^- \;, \qquad && [E_3^-, E_2^+] = -E_1^- \;, \\ &[E_3^+, E_3^-] = H_1 + H_2 \;. && \end{aligned} \tag{2.6}$$

The universal enveloping superalgebra $\mathcal{U} \equiv \mathcal{U}(sl(1|2))$ can be endowed with a classical super-Hopf structure, the $\mathbb{Z}_2$-graded coproduct $\underline{\Delta}$ being given by

$$\underline{\Delta}(X) = X \underline{\otimes} 1 + 1 \underline{\otimes} X \tag{2.7}$$

for $X \in sl(1|2)$ and extending super-multiplicatively to the entire $\mathcal{U}$. Here and in the following, we use an underlined notation for the graded structures. For example, $\underline{\otimes}$ denotes the $\mathbb{Z}_2$-graded tensor product satisfying

$$(A \underline{\otimes} B)(C \underline{\otimes} D) = (-1)^{\deg B . \deg C} (AC \underline{\otimes} BD) \;. \tag{2.8}$$

2.2 Casimir operators of $\mathcal{U}$

In this subsection, we focuse our attention on the centre of $\mathcal{U}$. We can construct a (countable) infinite set of Casimir operators $\mathcal{C}_p^{cl}$ where p is an integer ≥ 2. The explicit expression of $\mathcal{C}_p^{cl}$ is

$$\begin{aligned} \mathcal{C}_p^{cl} \;=\; & + H_1 H_2 \Big(H_1 - H_2\Big)^{p-2} \\ & - E_1^- E_1^+ \Big(H_2 (H_1 - H_2)^{p-2} + (1 - H_2)(H_1 - H_2 + 1)^{p-2}\Big) \\ & - E_2^- E_2^+ \Big(H_1 (H_1 - H_2)^{p-2} + (1 - H_1)(H_1 - H_2 - 1)^{p-2}\Big) \\ & - E_3^- E_3^+ \Big(H_1 - H_2\Big)^{p-2} \\ & + E_3^- E_2^+ E_1^+ \Big((H_1 - H_2)^{p-2} - (H_1 - H_2 + 1)^{p-2}\Big) \\ & + E_2^- E_1^- E_3^+ \Big((H_1 - H_2)^{p-2} - (H_1 - H_2 - 1)^{p-2}\Big) \\ & + E_2^- E_1^- E_2^+ E_1^+ \Big((H_1 - H_2 + 1)^{p-2} + (H_1 - H_2 - 1)^{p-2} - 2(H_1 - H_2)^{p-2}\Big) \;. \end{aligned} \tag{2.9}$$

They satisfy the following relations

$$\mathcal{C}_{p_1}^{cl} \mathcal{C}_{p_2}^{cl} = \mathcal{C}_{p_3}^{cl} \mathcal{C}_{p_4}^{cl} \quad \text{if} \quad p_1 + p_2 = p_3 + p_4 \quad \text{and} \quad p_i \geq 2 \;. \tag{2.10}$$

Nevertheless, it is important to note that none of these Casimir operators can be written as a polynomial function of the others. However, because of these relations, the eigenvalues of only two Casimir operators are enough to characterise a finite-dimensional highest weight irreducible representation of $sl(1|2)$.

Let h be the projection

$$h : \mathcal{U} = \mathcal{U}^0 \oplus \Big(\mathcal{U}\mathcal{U}^+ + \mathcal{U}^-\mathcal{U}\Big) \longrightarrow \mathcal{U}^0 \tag{2.11}$$

within the direct sum, where $\mathcal{U}^+$ and $\mathcal{U}^-$ are the subalgebras of $\mathcal{U}$ generated respectively by E_i^+ and E_i^- $(i = 1, 2)$, while $\mathcal{U}^0$ is the unital subalgebra generated by 1 and H_i $(i = 1, 2)$. The restriction $\bar{h}$ of h to the centre $\mathcal{Z}_{\mathcal{U}}$ of $\mathcal{U}$ is an algebra morphism onto the algebra ${\mathcal{U}^0}^W$ of polynomials in the Cartan generators H_1 and H_2 invariant under the action of the Weyl group, *i.e.* $(H_1 \leftrightarrow -H_2)^\dagger$. $\bar{h}$ is the Harish–Chandra homomorphism [7]. It is injective [13]. From Eq. (2.9), its image $\bar{h}\,(\mathcal{Z}_{\mathcal{U}})$ is $\{1\} \cup \mathcal{I}$, where $\mathcal{I}$ is the ideal in ${\mathcal{U}^0}^W$ generated by the product $H_1 H_2$. This result is compatible with the fact that the fields of fractions of ${\mathcal{U}^0}^W$ and $\bar{h}\,(\mathcal{Z}_{\mathcal{U}})$ coincide [7].

In reference [5], the formula

$$\mathcal{C}_p^{(m,n)} = \mathrm{STr}\left(\widehat{E}^{\;p}\right) \tag{2.12}$$

is given for the Casimir operators of the special linear superalgebras $sl(m|n)$. The matrix $\widehat{E}$ is defined by $\widehat{E}_B^A = (-1)^{\deg B} E_B^A$ $(A, B = 1, \cdots, m+n)$ where the matrix E collects the set of generators of $sl(m|n)$, with $\deg B = 1$ for $B = 1, \cdots, m$ and $\deg B = -1$ for $B = m+1, \cdots, m+n$. The authors of [5] proved that, in any finite-dimensional highest weight irreducible representation, the Casimir operators $\mathcal{C}_p^{(m,n)}$ with $p \geq m+n$ can be expressed in terms of the previous ones $\mathcal{C}_r^{(m,n)}$ with $r < p$.
Our Casimir operator $\mathcal{C}_p^{cl}$ of Eq. (2.9) is a linear combination of $\mathcal{C}_p^{(1,2)}$ and of a polynomial function in the $\mathcal{C}_r^{(1,2)}$ with $r < p$. It is actually *the* linear combination that eliminates all the higher degree terms in the raising and lowering generators $E_i^\pm$ and the image of which by the Harish–Chandra isomorphism is, up to a power of $H_1 - H_2$, a monomial of degree one in $H_1 H_2$. It is worthwhile to notice that in the case of $sl(n)$, this combination is precisely 0 for $p > n$, expressing the fact that $\mathcal{C}_p^n$ for $p > n$ is equal to a polynomial function in the $\mathcal{C}_r^n$, $r \leq n$. In the case of $sl(1|2)$ however, as operators, none of the Casimirs $\mathcal{C}_p^{cl}$ can be written in terms of the others.

2.3 Comments on the distinguished basis

Let us end this section with some comments on the distinguished basis. The generalised

†Note that the translated Weyl group coincides here with the Weyl group. Indeed, denoting by ρ_0 and ρ_1 the half sums of positive even and odd roots in the fermionic basis, the translation vector $\rho = \rho_0 - \rho_1$ vanishes for $sl(1|2)$.

Weyl transformation

$$\begin{array}{lll} h_1 = -H_1 - H_2 & h_2 = H_2 & \\ e_1^\pm = \pm E_3^\pm & e_2^\pm = E_2^\mp & e_3^\pm = \pm E_1^\pm \end{array} \tag{2.13}$$

defines the generators of the distinguished basis in terms of those of the fermionic basis. The relations in the Serre–Chevalley distinguished basis read

$$\begin{array}{l} [h_i, h_j] = 0 \quad \text{and} \quad [h_i, e_j^\pm] = \pm a'_{ij} e_j^\pm \ , \\ \{e_2^\pm, e_2^\pm\} = [e_1^\pm, e_2^\mp] = 0 \ , \\ [e_1^+, e_1^-] = h_1 \ , \\ \{e_2^+, e_2^-\} = h_2 \ , \end{array} \tag{2.14}$$

where (a'_{ij}) is the Cartan matrix in the distinguished basis:

$$(a'_{ij}) = \left(\begin{array}{cc} 2 & -1 \\ -1 & 0 \end{array} \right) \ . \tag{2.15}$$

All the results of this section obtained in the fermionic basis can be rewritten *mutatis mutandis* in the distinguished basis using the transformation Eq. (2.13).

3 The quantum superalgebra $\mathcal{U}_{qs}(sl(1|2))$

3.1 Presentation in the fermionic basis

We consider now the two-parameter quantum universal enveloping superalgebra $\mathcal{U}_{qs} \equiv \mathcal{U}_{qs}(sl(1|2))$. It is the two-parametric deformation of $\mathcal{U}(sl(1|2))$, defined by the generators H_i, $E_i^\pm$ $(i = 1, 2)$ and unit 1, and by the relations in the Serre–Chevalley fermionic basis:

$$\begin{array}{l} [H_i, H_j] = 0 \quad \text{and} \quad [H_i, E_j^\pm] = \pm a_{ij} E_j^\pm \ , \\ \{E_1^\pm, E_1^\pm\} = \{E_2^\pm, E_2^\pm\} = \{E_1^\pm, E_2^\mp\} = 0 \ , \\ \{E_1^+, E_1^-\} = [H_1]_q \, s^{-H_1} \ , \\ \{E_2^+, E_2^-\} = [H_2]_q \, s^{H_2} \ , \end{array} \tag{3.1}$$

with $[x]_q \equiv \dfrac{q^x - q^{-x}}{q - q^{-1}}$ (note that for $s = 1$, this is the standard definition of $\mathcal{U}_q(sl(1|2))$). Defining

$$\begin{array}{l} E_3^+ = qs^{-1} E_1^+ E_2^+ + E_2^+ E_1^+ \ , \\ E_3^- = E_1^- E_2^- + q^{-1} s^{-1} E_2^- E_1^- \ , \end{array} \tag{3.2}$$

the quantum Serre relations read

$$\begin{aligned}
&E_1^{\pm}E_3^{\pm} - q^{-1}s^{\pm 1}E_3^{\pm}E_1^{\pm} = 0 \ , \\
&E_2^{\pm}E_3^{\pm} - q\ s^{\mp 1}E_3^{\pm}E_2^{\pm} = 0 \ .
\end{aligned} \tag{3.3}$$

$\mathcal{U}_{qs}$ is equipped with the $\mathbb{Z}_2$-gradation given in Eq. (2.1) and is endowed with the structure of a super Hopf algebra by defining the coproduct $\underline{\Delta} : \mathcal{U}_{qs} \longrightarrow \mathcal{U}_{qs} \otimes \mathcal{U}_{qs}$, the antipode $\underline{S} : \mathcal{U}_{qs} \longrightarrow \mathcal{U}_{qs}$ and the counit $\underline{\epsilon} : \mathcal{U}_{qs} \longrightarrow \mathbb{C}$ as follows:

$$\begin{aligned}
&\underline{\Delta}(H_i) = H_i \underline{\otimes} 1 + 1 \underline{\otimes} H_i \ , \\
&\underline{\Delta}(E_1^+) = E_1^+ \underline{\otimes} 1 + q^{H_1}s^{-H_1} \underline{\otimes} E_1^+ \ , \\
&\underline{\Delta}(E_2^+) = E_2^+ \underline{\otimes} 1 + q^{H_2}s^{H_2} \underline{\otimes} E_2^+ \ , \\
&\underline{\Delta}(E_1^-) = E_1^- \underline{\otimes} q^{-H_1}s^{-H_1} + 1 \underline{\otimes} E_1^- \ , \\
&\underline{\Delta}(E_2^-) = E_2^- \underline{\otimes} q^{-H_2}s^{H_2} + 1 \underline{\otimes} E_2^- \ ,
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
&\underline{S}(H_i) = -H_i \ , \\
&\underline{S}(E_1^+) = -q^{-H_1}s^{H_1}E_1^+ \ , \quad \underline{S}(E_2^+) = -q^{-H_2}s^{-H_2}E_2^+ \ , \\
&\underline{S}(E_1^-) = -E_1^- q^{H_1}s^{H_1} \ , \quad \underline{S}(E_2^-) = -E_2^- q^{H_2}s^{-H_2} \ ,
\end{aligned} \tag{3.5}$$

$$\underline{\epsilon}(H_i) = \underline{\epsilon}(E_i^{\pm}) = 0 \ . \tag{3.6}$$

Notice that the antipode in Eq. (3.6) is super-antimultiplicative, *i.e.*

$$\underline{S}(AB) = (-1)^{\deg A . \deg B}\underline{S}(B)\underline{S}(A) \ . \tag{3.7}$$

We check by inspection that $S^2 = \mathrm{id}$.
It follows from Eqs. (3.1) that the remaining relations among the generators read

$$\begin{aligned}
&[E_3^+, E_1^-] = qsE_2^+q^{-H_1}s^{-H_1} \ , \\
&[E_3^+, E_2^-] = E_1^+q^{H_2}s^{H_2} \ , \\
&[E_3^-, E_1^+] = -E_2^- q^{H_1}s^{-H_1} \ , \\
&[E_3^-, E_2^+] = -q^{-1}sE_1^- q^{-H_2}s^{H_2} \ , \\
&[E_3^+, E_3^-] = [H_1 + H_2]_q \ s^{H_2 - H_1}
\end{aligned} \tag{3.8}$$

with the associated Hopf structure

$$\begin{aligned}
&\underline{\Delta}(E_3^+) = E_3^+ \underline{\otimes} 1 + \lambda s^{H_2}E_1^+q^{H_2} \underline{\otimes} E_2^+ + q^{H_2+H_1}s^{H_2-H_1} \underline{\otimes} E_3^+ \ , \\
&\underline{\Delta}(E_3^-) = E_3^- \underline{\otimes} q^{-H_2-H_1}s^{H_2-H_1} - \lambda E_2^- \underline{\otimes} q^{-H_2}E_1^- s^{H_2} + 1 \underline{\otimes} E_3^- \ ,
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
&\underline{S}(E_3^+) = -q^{-H_1-H_2}s^{H_1-H_2}E_3^+ + \lambda q^{-H_1-H_2}s^{H_1-H_2-1}E_1^+E_2^+ \ , \\
&\underline{S}(E_3^-) = -E_3^- q^{H_1+H_2}s^{H_1-H_2} - \lambda E_2^- E_1^- q^{H_1+H_2}s^{H_1-H_2-1} \ ,
\end{aligned} \tag{3.10}$$

$$\underline{\epsilon}(E_3^+) = \underline{\epsilon}(E_3^-) = 0 \ , \tag{3.11}$$

where $\lambda = q - q^{-1}$.

3.2 Casimir operators of $\mathcal{U}_{qs}$

One can compute the Casimir operators of $\mathcal{U}_{qs}$. As in the undeformed case, one finds by direct calculation a (countable) infinite set of Casimir operators $\mathcal{C}_p$, with p here in $\mathbb{Z}$, given by

$$\begin{aligned}\mathcal{C}_p = q^{(2p-1)(H_2-H_1)} \Big\{ & [H_1]_q[H_2]_q + E_1^- E_1^+ s^{H_1-1}\left(q^{-2p+1}[H_2-1]_q - [H_2]_q\right) \\ & + E_2^- E_2^+ s^{-H_2-1}\left(q^{2p-1}[H_1-1]_q - [H_1]_q\right) - q^{-1} E_3^- E_3^+ s^{H_1-H_2-1} \\ & - q^{p-2}(q-q^{-1})[p]_q E_2^- E_1^- E_3^+ s^{H_1-H_2-2} \\ & + q^{-p}(q-q^{-1})[p-1]_q E_3^- E_2^+ E_1^+ s^{H_1-H_2-1} \\ & + q^{-1}(q-q^{-1})^2[p]_q[p-1]_q E_2^- E_1^- E_2^+ E_1^+ s^{H_1-H_2-2} \Big\} .\end{aligned} \tag{3.12}$$

The Casimir operators $\mathcal{C}_p$ satisfy the following relations

$$\mathcal{C}_{p_1}\mathcal{C}_{p_2} = \mathcal{C}_{p_3}\mathcal{C}_{p_4} \quad \text{if} \quad p_1+p_2 = p_3+p_4 . \tag{3.13}$$

It is interesting to look at the classical limit of the $\mathcal{C}_p$'s. It is easy to see that $\lim_{q,s\to 1} \mathcal{C}_p = \mathcal{C}_2^{cl}$ for all $p \in \mathbb{Z}$. Moreover, the classical Casimir operators $\mathcal{C}_p^{cl}$ with $p \geq 3$ can be obtained as limits of suitable linear combinations of the quantum Casimir operators $\mathcal{C}_r$ involving coefficients with negative powers in $q - q^{-1}$ as follows

$$\mathcal{C}_p^{cl} = \lim_{q,s\to 1} \frac{1}{(q-q^{-1})^{p-2}} \sum_{l=0}^{p-2} (-1)^l \binom{p-2}{l} \mathcal{C}_l . \tag{$\simeq \pi$}$$

Let now h be the projection

$$h : \mathcal{U}_{qs} = {\mathcal{U}_{qs}}^0 \oplus \left(\mathcal{U}_{qs}{\mathcal{U}_{qs}}^+ + {\mathcal{U}_{qs}}^-\mathcal{U}_{qs}\right) \longrightarrow {\mathcal{U}_{qs}}^0 \tag{3.15}$$

within the direct sum, where ${\mathcal{U}_{qs}}^0$, ${\mathcal{U}_{qs}}^+$, ${\mathcal{U}_{qs}}^-$ are the subalgebras of $\mathcal{U}_{qs}$ generated respectively by $q^{\pm H_i}$ and $s^{\pm H_i}$, E_i^+, E_i^- $(i = 1, 2)$. The restriction $\bar{h}$ of h to the centre $\mathcal{Z}_{\mathcal{U}_{qs}}$ of $\mathcal{U}_{qs}$ is an algebra morphism onto the algebra of polynomials in the Cartan generators $q^{\pm 2H_1}$ and $q^{\pm 2H_2}$, invariant under the action of the Weyl group, *i.e.* $(H_1 \leftrightarrow -H_2)$. $\bar{h}$ is the quantum analogue of the Harish–Chandra homomorphism. From Eq. (3.12), its image $\bar{h}\left(\mathcal{Z}_{\mathcal{U}_{qs}}\right)$ is $\{1\} \cup \mathcal{I}_{qs}$, where $\mathcal{I}_{qs}$ is the ideal generated by the product $q^{H_1-H_2}[H_1]_q[H_2]_q$.

3.3 Presentation in the distinguished basis

As in the classical case, we can switch to the distinguished basis, the change of basis being given by

$$
\begin{array}{lll}
h_1 = -H_1 - H_2 & h_2 = H_2 & \\
e_1^+ = s^{-1} E_3^+ & e_2^+ = E_2^- q^{-H_2} s^{-H_2} & e_3^+ = s^{-1} E_1^+ \\
e_1^- = -E_3^- & e_2^- = s^{-1} E_2^+ q^{H_2} s^{-H_2} & e_3^- = -E_1^- \ .
\end{array}
\tag{3.16}
$$

This allows the derivation of the relations satisfied by the generators in the distinguished basis. The Casimir operators in the distinguished basis are hence provided from (3.12) using (3.16).
Unlike in the classical case, the natural Hopf structure in the distinguished basis is not the one provided by this redefinition. The natural coproduct of $\mathcal{U}_{qs}$ in the distinguished basis is actually given by (take $s = 1$ to see that it is natural)

$$
\begin{aligned}
&\underline{\tilde{\Delta}}(h_i) = h_i \underline{\otimes} 1 + 1 \underline{\otimes} h_i \ , \\
&\underline{\tilde{\Delta}}(e_1^+) = e_1^+ \underline{\otimes} 1 + q^{h_1} s^{h_1 + 2h_2} \underline{\otimes} e_1^+ \ , \\
&\underline{\tilde{\Delta}}(e_2^+) = e_2^+ \underline{\otimes} 1 + q^{h_2} s^{-h_2} \underline{\otimes} e_2^+ \ , \\
&\underline{\tilde{\Delta}}(e_1^-) = e_1^- \underline{\otimes} q^{-h_1} s^{h_1 + 2h_2} + 1 \underline{\otimes} e_1^- \ , \\
&\underline{\tilde{\Delta}}(e_2^-) = e_2^- \underline{\otimes} q^{-h_2} s^{-h_2} + 1 \underline{\otimes} e_2^- \ ,
\end{aligned}
\tag{3.17}
$$

where now

$$
\deg h_1 = \deg h_2 = \deg e_1^\pm = 0 \quad \text{and} \quad \deg e_2^\pm = \deg e_3^\pm = 1 \ . \tag{3.18}
$$

The above can be considered as an alternative Hopf structure compatible with the relations (3.1) defining the fermionic basis, in terms of which it reads

$$
\begin{aligned}
&\underline{\tilde{\Delta}}(H_i) = H_i \underline{\otimes} 1 + 1 \underline{\otimes} H_i \ , \\
&\underline{\tilde{\Delta}}(E_1^+) = E_1^+ \underline{\otimes} 1 + q^{-H_1} s^{-H_1} \underline{\otimes} E_1^+ - \lambda q^{H_2} E_3^+ s^{-H_2} \underline{\otimes} q^{-H_2} E_2^- s^{-H_2} \ , \\
&\underline{\tilde{\Delta}}(E_2^+) = E_2^+ \underline{\otimes} q^{-2H_2} + q^{-H_2} s^{H_2} \underline{\otimes} E_2^+ \ , \\
&\underline{\tilde{\Delta}}(E_1^-) = E_1^- \underline{\otimes} q^{H_1} s^{-H_1} + 1 \underline{\otimes} E_1^- - \lambda s^{-H_2} E_2^+ q^{H_2} \underline{\otimes} s^{-H_2} E_3^- q^{-H_2} \ , \\
&\underline{\tilde{\Delta}}(E_2^-) = E_2^- \underline{\otimes} q^{H_2} s^{H_2} + q^{2H_2} \underline{\otimes} E_2^- \ , \\
&\underline{\tilde{\Delta}}(E_3^+) = E_3^+ \underline{\otimes} 1 + q^{-H_1 - H_2} s^{H_2 - H_1} \underline{\otimes} E_3^+ \ , \\
&\underline{\tilde{\Delta}}(E_3^-) = E_3^- \underline{\otimes} q^{H_1 + H_2} s^{H_2 - H_1} + 1 \underline{\otimes} E_3^- \ .
\end{aligned}
\tag{3.19}
$$

In the construction of the Hamiltonians or for any other application, we can equivalently use the distinguished basis with its natural Hopf structure, or the fermionic basis with the new Hopf structure derived from the latter.

4 Super-FRT Construction

Anticipating the applications of section 6, we attempt now a formulation along the lines of the standard FRT construction. Our motivation is twofold: on the one hand, we aim at a significant computational simplification, due to the compactness of the notation used in [3], while, on the other hand, it is in this formulation that our results can most easily be generalised to the case of other quantum supergroups. We present our approach, as already outlined in the introduction, in two stages. In this section, continuing working in a $\mathbb{Z}_2$-graded environment, we find the analogues of the $L^{\pm}$ matrices of [3] in terms of which the algebraic and Hopf structures already presented in the Serre-Chevalley basis become particularly simple. We give furthermore the dual construction of the algebra and Hopf structure of the functions on $SL_{qs}(1|2)$ and comment on representations of $\mathbb{Z}_2$-graded tensor products. To further simplify calculations (by eliminating the need to keep track of super statistics), we apply, in the next section, the general theory of bosonization [9], thus obtaining a standard FRT-type Hopf algebra and its dual.

4.1 The deformed universal enveloping algebra

We introduce the 3×3 matrices $\underline{L}^+$, $\underline{L}^-$ (upper and lower triangular respectively), the elements of which, together with 1, generate $\mathcal{U}_{qs}$ and are given, in the Serre-Chevalley basis, by

$$\begin{array}{ll}
\underline{L}^+_{11} = q^{H_2} s^{H_2} & \underline{L}^-_{11} = q^{-H_2} s^{H_2} \\
\underline{L}^+_{22} = q^{H_2-H_1} s^{H_2+H_1} & \underline{L}^-_{22} = q^{-H_2+H_1} s^{H_2+H_1} \\
\underline{L}^+_{33} = q^{-H_1} s^{H_1} & \underline{L}^-_{33} = q^{H_1} s^{H_1} \\
\underline{L}^+_{12} = \lambda E^+_1 q^{H_2-H_1} s^{H_2+H_1} & \underline{L}^-_{21} = -q^{-1}\lambda E^-_1 q^{-H_2+H_1} s^{H_2+H_1} \\
\underline{L}^+_{23} = \lambda E^+_2 q^{-H_1} s^{H_1} & \underline{L}^-_{32} = q\lambda E^-_2 q^{H_1} s^{H_1} \\
\underline{L}^+_{13} = q\lambda E^+_3 q^{-H_1} s^{H_1} & \underline{L}^-_{31} = q\lambda E^-_3 q^{H_1} s^{H_1} \ .
\end{array} \tag{4.1}$$

The relations (3.1), (3.3) can now be expressed compactly as follows

$$R_{12}\underline{L}^{\pm}_2 \eta_{12} \underline{L}^{\pm}_1 = \underline{L}^{\pm}_1 \eta_{12} \underline{L}^{\pm}_2 R_{12}, \qquad R_{12}\underline{L}^{+}_2 \eta_{12} \underline{L}^{-}_1 = \underline{L}^{-}_1 \eta_{12} \underline{L}^{+}_2 R_{12}, \tag{4.2}$$

where

$$\eta_{ik,jl} = (-1)^{ik} \delta_{ij} \delta_{kl} = diag(-1, 1, -1, 1, 1, 1, -1, 1, -1) \tag{4.3}$$

and the $\mathcal{U}_{qs}$ R-matrix is given by

$$R = \left(\begin{array}{ccc|ccc|ccc} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q^{-1}s & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -q^{-1}s & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & -q^{-1}\lambda & 0 & q^{-1}s^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q^{-2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q^{-1}s & 0 & 0 & 0 \\ \hline 0 & 0 & -q^{-1}\lambda & 0 & 0 & 0 & -q^{-1}s^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -q^{-1}\lambda & 0 & q^{-1}s^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right) \tag{4.4}$$

with the matrix η as its classical limit. The explicit form of (4.2) can be found in the appendix. In the following we will make frequent use of the relations

$$\eta^2 = I, \quad \eta_{12} = \eta_{21}, \quad \eta R = R\eta, \quad R_{12}\eta_{13}\eta_{23} = \eta_{23}\eta_{13}R_{12}. \tag{4.5}$$

Starting from the product $\underline{L}_1^\pm \eta_{12} \underline{L}_2^\pm \eta_{13}\eta_{23}\underline{L}_3^\pm$ and using (4.2), in two different ways, to bring it to the form $\underline{L}_3^\pm \eta_{23}\eta_{13}\underline{L}_2^\pm \eta_{12}\underline{L}_1^\pm$ (according to the sequence of transpositions $123 \to 132 \to 312 \to 321$ and $123 \to 213 \to 231 \to 321$ respectively), one ensures that no cubic relations are imposed among the $\underline{L}_{ij}^\pm$'s if R satisfies the (ordinary) Quantum Yang-Baxter Equation (QYBE)

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \ ; \tag{4.6}$$

it is easily verified that the R given by (4.4) indeed satisfies this relation. In terms of the matrix $\hat{R}$, given by $\hat{R}_{ik,jl} = R_{ki,jl}$, (4.6) reads

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23}. \tag{4.7}$$

The $\hat{R}$ matrix is non-symmetric in our case and it satisfies the quadratic characteristic equation

$$\hat{R}^2 + q^{-1}\lambda\hat{R} - q^{-2} = 0 \ , \tag{4.8}$$

which implies the eigenvalues -1 and q^{-2}. With the fermionic degree of $\underline{L}_{ij}^\pm$ being given by

$$\deg \underline{L}_{ij}^\pm = i + j \pmod 2 \ , \tag{4.9}$$

we can express the superstatistics in matrix form as follows

$$\underline{L}_1^\pm \eta_{12} \underline{L}_2^{\pm\prime} \eta_{12} = \eta_{12} \underline{L}_2^{\pm\prime} \eta_{12} \underline{L}_1^\pm \ , \qquad \underline{L}_1^+ \eta_{12} \underline{L}_2^{-\prime} \eta_{12} = \eta_{12} \underline{L}_2^{-\prime} \eta_{12} \underline{L}_1^+ \ , \tag{4.10}$$

where $\underline{L}^\pm$, $\underline{L}^{\pm\prime}$ denote two copies of the generators living in different spaces in the tensor product $\mathcal{U}_{qs}\underline{\otimes}\mathcal{U}_{qs}$. The Hopf structure given in section 3 is now compactly encoded in the relations

$$\underline{\Delta}(\underline{L}^\pm) = \underline{L}^\pm \dot{\underline{\otimes}} \underline{L}^\pm \ , \qquad \underline{S}(\underline{L}^\pm) = (\underline{L}^\pm)^{-1}, \qquad \underline{\epsilon}(\underline{L}^\pm) = I \ , \tag{4.11}$$

where $\dot{\underline{\otimes}}$ stands for $\mathbb{Z}_2$-graded tensor product and matrix multiplication. Notice finally the superdeterminant relations (obtained by inspection of (4.1))

$$\underline{L}^{\pm}_{11}(\underline{L}^{\pm}_{22})^{-1}\underline{L}^{\pm}_{33} = 1 \ . \tag{4.12}$$

One can combine all the generators in a single matrix $\underline{Y}$, given by [3]

$$\underline{Y} = \underline{L}^{+}\underline{S}(\underline{L}^{-}) \ , \tag{4.13}$$

the elements of which are easily seen to satisfy the commutation relations

$$R_{21}\underline{Y}_1 R_{12}\underline{Y}_2 = \underline{Y}_2 R_{21}\underline{Y}_1 R_{12} \ , \tag{4.14}$$

with $\deg \underline{Y}_{ij} = i + j \pmod 2$.

4.2 The dual superalgebra of functions

4.2.1 The construction of the dual

The algebra $\mathcal{A}_{qs} = \mathrm{Fun}(SL_{qs}(1|2))$, dual to $\mathcal{U}_{qs}$, is generated by 1 and the elements of the (3×3, in our case) supermatrix $\underline{A}$ – their inner product with $\underline{L}^{\pm}$ is given by

$$\left\langle\!\!\left\langle \underline{L}^{+}_1, \underline{A}_2 \right\rangle\!\!\right\rangle = \eta_{12} R_{21} \ , \qquad \left\langle\!\!\left\langle \underline{L}^{-}_1, \underline{A}_2 \right\rangle\!\!\right\rangle = \eta_{12} R^{-1}_{12} \ , \qquad \left\langle\!\!\left\langle 1, \underline{A} \right\rangle\!\!\right\rangle = I \ . \tag{4.15}$$

The $\mathbb{Z}_2$-graded Hopf structure is

$$\underline{\Delta}(\underline{A}) = \underline{A}\dot{\underline{\otimes}}\underline{A} \ , \qquad \underline{S}(\underline{A}) = \underline{A}^{-1} \ , \qquad \underline{\epsilon}(\underline{A}) = I \ , \tag{4.16}$$

with the coproduct being related to the product in the dual via (x, $y \in \mathcal{U}_{qs}$)

$$\left\langle\!\!\left\langle xy, \underline{A}_{ij} \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle x, \underline{A}_{im} \right\rangle\!\!\right\rangle \left\langle\!\!\left\langle y, \underline{A}_{mj} \right\rangle\!\!\right\rangle \ . \tag{4.17}$$

The reader will notice here that although going from the lhs to the rhs of (4.17) involves (typographicaly) a transposition of y and $\underline{A}_{im}$, no corresponding sign factor was included on the rhs. The advantage of this convention is that it supplies us with a matrix representation of $\mathcal{U}_{qs}$ (with the usual matrix multiplication), a feature that we find especially appealing in view of the applications of section 6. On the other hand, viewing this quantum supergroup as a particular example of a braided group (see, for example, [9] and references therein), one would probably want to attempt a formulation that is compatible with the diagrammatics of category theory. We give, for completeness, such a formulation at the end of this section.

We check now, in order to illustrate the formalism, whether the first of (4.16) is compatible with the "RLL" commutation relations, Eq. (4.2). The inner product of the lhs of (4.2) with $\underline{A}$ is given by

$$\begin{aligned} \left\langle\!\!\left\langle R_{23}\underline{L}^{+}_3 \eta_{23}\underline{L}^{+}_2, \underline{A}_1 \right\rangle\!\!\right\rangle &= R_{23} \left\langle\!\!\left\langle \underline{L}^{+}_3, \underline{A}_1 \right\rangle\!\!\right\rangle \eta_{23} \left\langle\!\!\left\langle \underline{L}^{+}_2, \underline{A}_1 \right\rangle\!\!\right\rangle \\ &= R_{23}\eta_{13}R_{13}\eta_{23}\eta_{12}R_{12} \\ &= R_{23}R_{13}\eta_{13}\eta_{23}R_{12}\eta_{12} \\ &= R_{23}R_{13}R_{12}\eta_{12}\eta_{13}\eta_{23} \ , \end{aligned}$$

while for the right hand side we get

$$\begin{aligned}
\left\langle\!\!\left\langle \underline{L}_2^+ \eta_{23} \underline{L}_3^+ R_{23}, \underline{A}_1 \right\rangle\!\!\right\rangle &= \left\langle\!\!\left\langle \underline{L}_2^+, \underline{A}_1 \right\rangle\!\!\right\rangle \eta_{23} \left\langle\!\!\left\langle \underline{L}_3^+, \underline{A}_1 \right\rangle\!\!\right\rangle R_{23} \\
&= \eta_{12} R_{12} \eta_{23} \eta_{13} R_{13} R_{23} \\
&= R_{12} \eta_{12} \eta_{23} R_{13} \eta_{13} R_{23} \\
&= R_{12} R_{13} \eta_{12} \eta_{13} \eta_{23} R_{23} \\
&= R_{12} R_{13} R_{23} \eta_{12} \eta_{13} \eta_{23}
\end{aligned}$$

and, by invoking the QYBE for R, one verifies that $\underline{A}$ is a representation. The degree of its elements is given by

$$\deg \underline{A}_{ij} = (-1)^{i+j} \quad (\bmod 2) \; ; \tag{4.18}$$

(this is the standard fermionic basis format - the block format is standard in the distinguished basis). Notice that the requirement that the quantum superdeterminants of Eq. (4.12) be represented by the unit matrix fixes the normalization of the R-matrix.

4.2.2 Tensor product representations

We address next the question of tensor product representations. It is clear that the matrix $\underline{A}_1 \underline{\otimes} \underline{A}_2$ does not provide a representation for $\mathcal{U}_{qs} \underline{\otimes} \mathcal{U}_{qs}$ since, for example, the elements $\underline{L}^+ \underline{\otimes} 1$ and $1 \underline{\otimes} \underline{L}^+$ do not commute (see Eq. (4.10)), while their inner products with $\underline{A}_1 \underline{\otimes} \underline{A}_2$ necessarily do. An alternative statement of this fact can be made via the naturally induced coproduct $\underline{\Delta}^{\{2\}}$ in $\mathcal{A}_{qs} \underline{\otimes} \mathcal{A}_{qs}$

$$\underline{\Delta}^{\{2\}}(a \underline{\otimes} b) = (-1)^{\deg b_{(\underline{1})} \, \deg a_{(\underline{2})}} \; (a_{(\underline{1})} \underline{\otimes} b_{(\underline{1})}) \underline{\otimes} (a_{(\underline{2})} \underline{\otimes} b_{(\underline{2})}) \tag{4.19}$$

where (Sweedler notation)

$$\underline{\Delta}(a) \equiv \sum_i a^i_{(\underline{1})} \underline{\otimes} a^i_{(\underline{2})} \equiv a_{(\underline{1})} \underline{\otimes} a_{(\underline{2})} \; . \tag{4.20}$$

The failure of $\underline{A}_1 \underline{\otimes} \underline{A}_2$ to provide a representation for $\mathcal{U}_{qs} \underline{\otimes} \mathcal{U}_{qs}$ can be traced to the fact that

$$\underline{\Delta}^{\{2\}}(\underline{A}_1 \underline{\otimes} \underline{A}_2) \neq (\underline{A}_1 \underline{\otimes} \underline{A}_2) \underline{\otimes} (\underline{A}_1 \underline{\otimes} \underline{A}_2) \; . \tag{4.21}$$

We find though that the matrix

$$\underline{A}_{12} \equiv \underline{A}_1 \underline{\otimes} \eta_{12} \underline{A}_2 \eta_{12} \tag{4.22}$$

satisfies

$$\underline{\Delta}^{\{2\}}(\underline{A}_{12}) = \underline{A}_{12} \underline{\otimes} \underline{A}_{12} \; . \tag{4.23}$$

It is clear that $\underline{A}_{12}$ is a representation for $\mathcal{U}_{qs}\underline{\otimes}1$ and $1\underline{\otimes}\mathcal{U}_{qs}$ (the latter since $\eta^2 = I$). We check whether it represents the "cross" commutation relations, Eq. (4.10). For the inner product of the lhs of the first of (4.10) with $\underline{A}_{34}$ we get

$$\begin{aligned}
\left\langle\!\!\left\langle (\underline{L}_1^+\underline{\otimes}1)\eta_{12}(1\underline{\otimes}\underline{L}_2^+)\eta_{12}, \underline{A}_{34} \right\rangle\!\!\right\rangle &= \left\langle\!\!\left\langle \underline{L}_1^+\underline{\otimes}1, \underline{A}_{34} \right\rangle\!\!\right\rangle \eta_{12} \left\langle\!\!\left\langle 1\underline{\otimes}\underline{L}_2^+, \underline{A}_{34} \right\rangle\!\!\right\rangle \eta_{12} \\
&= R_{31}\eta_{34}^2\eta_{12}\eta_{34}R_{42}\eta_{34}\eta_{12} \\
&= R_{31}\eta_{12}\eta_{23}^2\eta_{34}R_{42}\eta_{34}\eta_{12} \\
&= \eta_{12}\eta_{23}R_{31}R_{42}\eta_{23}\eta_{12}
\end{aligned}$$

while the rhs gives

$$\begin{aligned}
\left\langle\!\!\left\langle \eta_{12}(1\underline{\otimes}\underline{L}_2^+)\eta_{12}(\underline{L}_1^+\underline{\otimes}1), \underline{A}_{34} \right\rangle\!\!\right\rangle &= \eta_{12} \left\langle\!\!\left\langle 1\underline{\otimes}\underline{L}_2^+, \underline{A}_{34} \right\rangle\!\!\right\rangle \eta_{12} \left\langle\!\!\left\langle \underline{L}_1^+\underline{\otimes}1, \underline{A}_{34} \right\rangle\!\!\right\rangle \\
&= \eta_{12}\eta_{34}R_{42}\eta_{34}\eta_{12}R_{31} \\
&= \eta_{12}\eta_{34}R_{42}\eta_{34}\eta_{23}\eta_{23}\eta_{12}R_{31} \\
&= \eta_{12}\eta_{34}\eta_{34}\eta_{23}R_{42}R_{31}\eta_{23}\eta_{12} \\
&= \eta_{12}\eta_{23}R_{42}R_{31}\eta_{23}\eta_{12} \ ;
\end{aligned}$$

we conclude that $\underline{A}_{34}$ provides a representation for $\mathcal{U}_{qs}\underline{\otimes}\mathcal{U}_{qs}$. The extension to higher tensor products gives the representation matrix $\underline{A}_{1\ldots L}$ defined by

$$\underline{A}_{1\ldots L} = \underline{\bigotimes_{k=1}^{L}} \underline{A}_{(k)} \tag{4.24}$$

where

$$\underline{A}_{(k)} \equiv \eta_{1k}\ldots\eta_{k-1,k}\underline{A}_k\eta_{k-1,k}\ldots\eta_{1k} \tag{4.25}$$

which satisfies

$$\underline{\Delta}^{\{L\}}(\underline{A}_{1\ldots L}) = \underline{A}_{1\ldots L}\underline{\otimes}\underline{A}_{1\ldots L} \ , \tag{4.26}$$

$\underline{\Delta}^{\{L\}}$ being the naturally induced coproduct in $\mathcal{A}_{qs}^{\underline{\otimes}L}$.

A number of consistency checks are in order at this point. We verify, as an example, the compatibility of the $\underline{A}-\underline{A}$ commutation relations with the $\underline{A}-\underline{A}'$ superstatistics (given by an equation analogous to (4.10))

$$\begin{aligned}
(R_{12}\underline{A}_1\eta_{12}\underline{A}_2 - \underline{A}_2\eta_{12}\underline{A}_1R_{12})\eta_{13}\eta_{23}\underline{A}_3' &= R_{12}\underline{A}_1\eta_{12}\eta_{13}\eta_{23}\underline{A}_3'\eta_{23}\underline{A}_2\eta_{23} \\
&\quad -\underline{A}_2\eta_{12}\eta_{23}\eta_{13}\underline{A}_3'\eta_{13}\underline{A}_1\eta_{13}R_{12} \\
&= R_{12}\eta_{23}\eta_{13}\underline{A}_3'\eta_{13}\underline{A}_1\eta_{13}\eta_{12}\eta_{23}\underline{A}_2\eta_{23} \\
&\quad -\eta_{13}\eta_{23}\underline{A}_3'\eta_{23}\underline{A}_2\eta_{23}\eta_{12}\eta_{13}\underline{A}_1\eta_{13}R_{12} \\
&= \eta_{23}\eta_{13}\underline{A}_3'R_{12}\eta_{13}\eta_{23}\underline{A}_1\eta_{12}\underline{A}_2\eta_{13}\eta_{23} \\
&\quad -\eta_{13}\eta_{23}\underline{A}_3'\eta_{23}\eta_{13}\underline{A}_2\eta_{12}\underline{A}_1R_{12}\eta_{13}\eta_{23} \\
&= \eta_{13}\eta_{23}\underline{A}_3'\eta_{13}\eta_{23}(R_{12}\underline{A}_1\eta_{12}\underline{A}_2 - \underline{A}_2\eta_{12}\underline{A}_1R_{12})\eta_{13}\eta_{23} \ .
\end{aligned}$$

One can show, along the same lines, the consistency of the entire scheme presented above.

4.2.3 $\mathbb{Z}_2$-graded inner product

As promised earlier in this section, we present now a version of the construction of $\mathcal{A}_{qs}$ that uses the $\mathbb{Z}_2$-graded inner product $\llbracket \cdot, \cdot \rrbracket$ defined by

$$\llbracket \underline{L}_1^+, \eta_{12}\underline{A}_2 \rrbracket = R_{21} \ , \qquad \llbracket \underline{L}_1^-, \eta_{12}\underline{A}_2 \rrbracket = R_{12}^{-1} \ , \qquad \llbracket 1, \underline{A} \rrbracket = I \tag{4.27}$$

and the duality relations (x, $y \in \mathcal{U}_{qs}$, a, $b \in \mathcal{A}_{qs}$, with y, a of homogeneous degree)

$$\begin{aligned} \llbracket xy, \underline{A}_{ij} \rrbracket &= \llbracket x \underline{\otimes} y, \underline{A}_{im} \underline{\otimes} \underline{A}_{mj} \rrbracket \\ &\equiv (-1)^{\deg \underline{A}_{im} \cdot \deg y} \llbracket x, \underline{A}_{im} \rrbracket \llbracket y, \underline{A}_{mj} \rrbracket \ , \end{aligned} \tag{4.28}$$

$$\begin{aligned} \llbracket \underline{L}_{ij}^\pm, ab \rrbracket &= \llbracket \underline{L}_{im}^\pm \underline{\otimes} \underline{L}_{mj}^\pm, a \underline{\otimes} b \rrbracket \\ &\equiv (-1)^{\deg \underline{L}_{mj}^\pm \cdot \deg a} \llbracket \underline{L}_{im}^\pm, a \rrbracket \llbracket \underline{L}_{mj}^\pm, b \rrbracket \ . \end{aligned} \tag{4.29}$$

$\mathcal{A}_{qs}$ remains unchanged as a Hopf algebra. One can go further and give the natural semidirect (graded) product commutation relations between $\underline{L}^\pm$ and $\underline{A}$

$$\underline{L}_1^+ \eta_{12} \underline{A}_2 = \eta_{12} \underline{A}_2 R_{21} \underline{L}_1^+ \eta_{12} \tag{4.30}$$

$$\underline{L}_1^- \eta_{12} \underline{A}_2 = \eta_{12} \underline{A}_2 R_{12}^{-1} \underline{L}_1^- \eta_{12} \ . \tag{4.31}$$

5 Bosonization

5.1 A bosonised basis

We start with the following observation: the element $g = e^{i\pi(H_1+H_2)}$ of $\mathcal{U}_{qs}$ satisfies

$$g\,x = (-1)^{\deg x} x\,g \tag{5.1}$$

where x is any element (of homogeneous degree) of $\mathcal{U}_{qs}$. Notice also that $\underline{\Delta}(g) = g \underline{\otimes} g$ and

$$\langle\!\langle g, \underline{A} \rangle\!\rangle = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{5.2}$$

so that one can consistently set $g^2 = 1$ as an operator relation in the algebra. Under these conditions, one may apply the general construction of bosonization of a super-Hopf algebra [9] to obtain an ordinary (*i.e.* non-$\mathbb{Z}_2$-graded) coproduct for the $\underline{L}_{ij}^\pm$'s

$$\Delta(\underline{L}_{ij}^\pm) = \sum_m \underline{L}_{im}^\pm g^{m+j} \otimes \underline{L}_{mj}^\pm \ , \tag{5.3}$$

along with a matching counit and antipode. The entire bosonic Hopf structure is most compactly expressed in terms of the matrices $L^{\pm}$ given by

$$L^{\pm} = \underline{L}^{\pm} G \ , \qquad G \equiv \ \mathrm{diag}\,(g \ , 1 \ , g) \ . \tag{5.4}$$

Then (5.3) and the rest of the Hopf structure become

$$\Delta(L^{\pm}) = L^{\pm}\dot{\otimes}L^{\pm} \ , \qquad S(L^{\pm}) = L^{\pm^{-1}} \ , \qquad \epsilon(L^{\pm}) = I \ . \tag{5.5}$$

Starting from (4.2), one finds for the $L-L$ commutation relations the standard FRT expressions

$$R_{12}L_2^{\pm}L_1^{\pm} = L_1^{\pm}L_2^{\pm}R_{12} \ , \qquad R_{12}L_2^{+}L_1^{-} = L_1^{-}L_2^{+}R_{12} \ . \tag{5.6}$$

Notice that (5.6) just expresses the original $\mathcal{U}_{qs}$ algebra in a still different basis while the Hopf structure of (5.5) is not equivalent to the ones presented in the previous two sections. The explicit form of (5.6) can be found in the appendix. For the square of the antipode we find, by direct calculation,

$$S^2(L^{\pm}) = gL^{\pm}g^{-1} = DL^{\pm}D^{-1} \tag{5.7}$$

where $D = \ \mathrm{diag}\,(-1 \ ,1 \ ,-1)$. Since $D^2 = I$, we have $S^4 = \mathrm{id}$. The D matrix further satisfies

$$\mathrm{Tr}_2(D_2\hat{R}_{12}) = I_1 \ , \qquad \mathrm{Tr}_1(D_1^{-1}\hat{R}_{12}^{-1}) = I_2 \ , \qquad D_1(R^{T_2})_{12}^{-1} = (R^{-1})_{12}^{T_2}D_1 \ , \tag{5.8}$$

where T_2 denotes transposition in the second matrix space. Using the characteristic equation for $\hat{R}$, one can derive from (5.8) two more useful identities

$$\mathrm{Tr}_2(D_2\hat{R}_{12}^{-1}) = I_1 \ , \qquad \mathrm{Tr}_1(D_1^{-1}\hat{R}_{12}) = I_2 \ . \tag{5.9}$$

Finally, the Y matrix, defined by $Y = L^{+}S(L^{-})$, is found to be equal to $\underline{Y}$.

The construction of the dual proceeds as in [3] – the formulas for the commutation relations and the Hopf structure can be obtained from those of the previous section by setting $\eta = I$ and omitting the underlining of symbols. The inner product relations are

$$\left\langle L_1^{+}, A_2 \right\rangle = R_{21} \ , \qquad \left\langle L_1^{-}, A_2 \right\rangle = R_{12}^{-1} \ , \qquad \left\langle 1, A \right\rangle = I \ . \tag{5.10}$$

Notice the use of a (still) different symbol for the inner product. The ones of the previous section relate, via equations (4.17), (4.27), products in $\mathcal{U}_{qs}$ with graded coproducts in $\mathcal{A}_{qs}$ while the one appearing in (5.10) conforms to the (standard) Hopf algebra duality requirement

$$\left\langle xy, a \right\rangle = \left\langle x, a_{(1)} \right\rangle \left\langle y, a_{(2)} \right\rangle \ , \qquad \Delta a \equiv a_{(1)} \otimes a_{(2)} \ . \tag{5.11}$$

5.2 The centre of $\mathcal{U}_{qs}$

We now turn our attention to the centre of $\mathcal{U}_{qs}$. We consider the set of Casimirs given by

$$c^{(k)} = \lambda^{-k}\,\mathrm{Tr}(D^{-1}(I-Y)^k) \qquad k \in \mathbb{Z} \tag{5.12}$$

(these are linearly related to the Casimirs introduced in [3]). The proof of their centrality relies on the third of (5.8). With the representation of Y being given by

$$\left\langle Y_1, A_2 \right\rangle = \hat{R}_{12}^2 \tag{5.13}$$

and the characteristic equation (4.8) allowing us to write

$$(I-\hat{R}^2)^k = \frac{(1-q^{-4})^k}{1+q^{-2}}(\hat{R}+I)\ , \tag{5.14}$$

we find for the values of the casimirs in the fundamental representation

$$\begin{aligned} \left\langle c^{(k)}, A_2 \right\rangle &= \mathrm{Tr}_1(D_1^{-1}\left\langle (I_1-Y_1))^k, A_2 \right\rangle) \\ &= \mathrm{Tr}_1(D_1^{-1}(I_{12}-\hat{R}_{12}^2)^k) \\ &= \frac{(1-q^{-4})^k}{1+q^{-2}}\,\mathrm{Tr}_1(D_1^{-1}(\hat{R}_{12}+I_{12})) \\ \Rightarrow \left\langle c^{(k)}, A_2 \right\rangle &= 0\ . \end{aligned} \tag{5.15}$$

The construction of the Hamiltonians of the next section requires the evaluation of the representation of the coproduct of central elements in $\mathcal{U}_{qs}$. For the above set of Casimirs (and $k \geq 1$) we compute

$$\begin{aligned} c_{23}^{(k)} &\equiv \left\langle \Delta(c^{(k)}), A_2 \otimes A_3 \right\rangle \\ &= \mathrm{Tr}_1 D_1^{-1}\lambda^{-k}\left\langle (\Delta(I_1-Y_1))^k, A_2\otimes A_3 \right\rangle \\ &= \mathrm{Tr}_1 D_1^{-1} X_{123}^k \end{aligned} \tag{5.16}$$

where

$$\begin{aligned} X_{123} &\equiv \lambda^{-1}\left\langle \Delta(I_1-Y_1), A_2\otimes A_3 \right\rangle \\ &= q^{-1}\hat{R}_{23}\hat{R}_{12}\hat{R}_{23} + q^{-3}\hat{R}_{12} + q^{-1} + q^{-3}\ . \end{aligned} \tag{5.17}$$

To extract more information about $c_{23}^{(k)}$, we assume that, for some k, X_{123}^k is of the form

$$X_{123}^k = a_k\hat{R}_{23}\hat{R}_{12}\hat{R}_{23} + b_k(\hat{R}_{23}\hat{R}_{12}+\hat{R}_{12}\hat{R}_{23}) + c_k\hat{R}_{12} + d_k\hat{R}_{23} + f_k I \tag{5.18}$$

and, furthermore, that for that same k, the relation

$$c_k = q^{-2}a_k + (1-q^{-2})b_k \tag{5.19}$$

holds (both assumptions are evidently valid for $k=1$). Then, multiplying (5.18) by X_{123} and using the characteristic equation for $\hat{R}$, we find the recursion relations

$$\begin{aligned}
a_{k+1} &= (q^{-7}-q^{-5}+2q^{-3})a_k + (2q^{-5}-3q^{-3}+2q^{-1})b_k + (q^{-3}-q^{-1})c_k \\
&\quad +(q^{-3}-q^{-1})d_k + q^{-1}f_k \\
b_{k+1} &= (q^{-7}-q^{-5}+q^{-3})a_k + (2q^{-5}-q^{-3}+q^{-1})b_k + q^{-3}c_k + q^{-3}d_k \\
c_{k+1} &= (q^{-7}-q^{-5})a_k + q^{-5}b_k + (q^{-5}+q^{-1})c_k + q^{-3}f_k \\
d_{k+1} &= (q^{-7}-q^{-5})a_k + 2q^{-5}b_k + (q^{-3}+q^{-1})d_k \\
f_{k+1} &= q^{-7}a_k + q^{-5}c_k + (q^{-3}+q^{-1})f_k
\end{aligned} \tag{5.20}$$

from which it easily follows, by induction, that both (5.18) and (5.19) are valid for all $k \geq 1$. Taking the quantum trace on both sides of (5.18) we find, with the help of the second of (5.8)

$$c_{23}^{(k)} = (-q^{-1}\lambda a_k + 2b_k - d_k)\hat{R}_{23} + (q^{-2}a_k + c_k - f_k)I \ . \tag{5.21}$$

We assume that, for some k, the above expression is proportional to $\hat{R}_{23} + I$ (true for $k=1$). This gives us the relation

$$f_k = (1+q^{-2})a_k - (1+q^{-2})b_k + d_k \tag{5.22}$$

which is easily established, by induction, for all $k \geq 1$. Implementing (5.19), (5.22) in (5.20) we are left with a recursion relation for the sequences a_k, b_k, d_k – solving it and substituting in (5.21) we finally find

$$c_{23}^{(k)} = \alpha_k(\hat{R}+I)_{23} \ , \qquad \alpha_k = \frac{q^{-4k+1}}{[2]_q[3]_q}([4]_q^k - q^{3k}[2]_q^2) \ . \tag{5.23}$$

6 Hamiltonians

6.1 General scheme

In this section, we apply the results concerning the Casimir operators to construct 3^L-state quantum chain Hamiltonians with nearest neighbour interaction that are $sl(1|2)$ (resp. $\mathcal{U}_{qs}(sl(1|2))$) invariant.

Define a L-site Hamiltonian $\mathcal{H}_p^{(1\cdots L)}$ by

$$\mathcal{H}_p^{(1\cdots L)} = \sum_{j=1}^{L-1} \mathcal{H}_p^{(j,j+1)} \tag{6.1}$$

where

$$\mathcal{H}_p^{(j,j+1)} = 1 \otimes \cdots \otimes \mathcal{H}_p \otimes \cdots \otimes 1 \tag{6.2}$$

with the two-site Hamiltonian $\mathcal{H}_p$ in position $(j, j+1)$.
The two-site Hamiltonian $\mathcal{H}_p$ itself is defined by

$$(\mathcal{H}_p)_{12} = \left\langle\!\!\left\langle \underline{\Delta}(\mathcal{C}_p), \underline{A}_{12} \right\rangle\!\!\right\rangle \tag{6.3}$$

for any Casimir operator $\mathcal{C}_p$ of $\mathcal{U}$ (resp. $\mathcal{U}_{qs}$).
Then the L-site Hamiltonian $\mathcal{H}_p^{(1\cdots L)}$ is $\mathcal{U}$-invariant (resp. $\mathcal{U}_{qs}$-invariant) in the sense that

$$\left[\mathcal{H}_p^{(1\cdots L)}, X_{1\cdots L}\right] = 0 \tag{6.4}$$

where $X_{1\cdots L}$ is the evaluation in the L-fold tensor product of the fundamental representation of $X \in \mathcal{U}$ (resp. $\mathcal{U}_{qs}$) *i.e.*

$$X_{1\cdots L} \equiv \left\langle\!\!\left\langle \underline{\Delta}^{(L-1)}(X), \underline{A}_{1\cdots L} \right\rangle\!\!\right\rangle \ . \tag{6.5}$$

6.2 The classical case

The fundamental representation of $sl(1|2)$ is given by

$$\begin{array}{lll}
\pi(H_1) = e_{11} + e_{22} \ , & \pi(H_2) = -e_{22} - e_{33} \ , & \\
\pi(E_1^+) = e_{21} \ , & \pi(E_2^+) = e_{32} \ , & \pi(E_3^+) = e_{31} \ , \\
\pi(E_1^-) = e_{12} \ , & \pi(E_2^-) = -e_{23} \ , & \pi(E_3^-) = -e_{13} \ ,
\end{array} \tag{6.6}$$

where e_{ij} are 3×3 elementary matrices with entry 1 in ith row and jth column and 0 elsewhere.
By applying the general scheme above to the classical Casimir operators $\mathcal{C}_p^{cl}$ computed in Section 2, one finds that the corresponding Hamiltonians $\mathcal{H}_p^{cl}$ are of the form $\mathcal{H}_p^{cl} = \alpha_p \mathcal{H}^{cl} + \beta_p I$ where I is the identity and $\mathcal{H}^{cl}$ is given by

$$\mathcal{H}^{cl} = \left(\begin{array}{ccc|ccc|ccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\
\hline
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
\hline
0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right) \ . \tag{6.7}$$

To get a physical interpretation of $\mathcal{H}^{cl}$, one can use the realization of $sl(1|2)$ in terms of fermionic creation and annihilation operators $c_{\uparrow i}^{\pm}$ and $c_{\downarrow i}^{\pm}$ such that

$$\{c_{\alpha i}^{-}, c_{\beta j}^{+}\} = \delta_{\alpha\beta}\delta_{ij} \tag{6.8}$$

where $\alpha, \beta = \uparrow$ or $\downarrow$ and i, j are the site indices.
Let us define the spin operators $\sigma_j^+, \sigma_j^-, \sigma_j^0$ by

$$\sigma_j^+ = c_{\uparrow j}^+ c_{\downarrow j}^- \ , \quad \sigma_j^- = c_{\downarrow j}^+ c_{\uparrow j}^- \ , \quad \sigma_j^0 = c_{\uparrow j}^+ c_{\uparrow j}^- - c_{\downarrow j}^+ c_{\downarrow j}^- \ , \tag{6.9}$$

and the number operators $n_{\uparrow j}$, $n_{\downarrow j}$, n_j, $n_{\emptyset j}$ by

$$n_{\uparrow j} = c_{\uparrow j}^+ c_{\uparrow j}^- \ , \quad n_{\downarrow j} = c_{\downarrow j}^+ c_{\downarrow j}^- \ , \quad n_j = n_{\uparrow j} + n_{\downarrow j} \ , \quad n_{\emptyset j} = 1 - n_j \ . \tag{6.10}$$

Then the realization of the superalgebra $sl(1|2)$ is given by

$$\begin{array}{lll} e_{11} = n_\uparrow \ , & e_{22} = n_\emptyset \ , & e_{33} = n_\downarrow \ , \\ e_{12} = c_\uparrow^+ (1 - n_\downarrow) \ , & e_{23} = c_\downarrow^- (1 - n_\uparrow) \ , & e_{13} = \sigma^+ \ , \\ e_{21} = c_\uparrow^- (1 - n_\downarrow) \ , & e_{32} = c_\downarrow^+ (1 - n_\uparrow) \ , & e_{31} = \sigma^- \ . \end{array} \tag{6.11}$$

It follows that the two-site Hamiltonian $\mathcal{H}^{cl\,(j,j+1)}$ can be expressed as

$$\begin{aligned} \mathcal{H}^{cl\,(j,j+1)} \ = \ & + c_{\uparrow j}^+ (1 - n_{\downarrow j}) c_{\uparrow j+1}^- (1 - n_{\downarrow j+1}) - c_{\uparrow j}^- (1 - n_{\downarrow j}) c_{\uparrow j+1}^+ (1 - n_{\downarrow j+1}) \\ & + c_{\downarrow j}^+ (1 - n_{\uparrow j}) c_{\downarrow j+1}^- (1 - n_{\uparrow j+1}) - c_{\downarrow j}^- (1 - n_{\uparrow j}) c_{\downarrow j+1}^+ (1 - n_{\uparrow j+1}) \\ & - \sigma_j^+ \sigma_{j+1}^- - \sigma_j^- \sigma_{j+1}^+ \\ & + (1 - n_{\downarrow j})(1 - n_{\uparrow j+1}) + (1 - n_{\uparrow j})(1 - n_{\downarrow j+1}) \ . \end{aligned} \tag{6.12}$$

which is precisely the Hamiltonian that describes the $t - J$ model at the supersymmetric point. This model was found to be invariant under the $sl(1|2)$ superalgebra [16].

6.3 The deformed case

6.3.1 $\mathcal{U}_{qs}$-invariant Hamiltonians from the fermionic basis

The fundamental representation of $\mathcal{U}_{qs}$ remains, as usual, undeformed and is given by (6.6). Using the FRT formalism and the set of Casimir operators $c^{(k)}$ of Sect. 5, one gets $\mathcal{H}_{12} = c_{12}^{(k)}$, see Eq. (5.23). One can now compute the Hamiltonians $\mathcal{H}_p$ associated to the Casimir operators $\mathcal{C}_p$. The result is

$$\mathcal{H}_p = -q^{3-6p}(q - q^{-1})^2 \mathcal{H}_{ferm} \ , \tag{6.13}$$

where

$$\mathcal{H}_{ferm} = \left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q^{-1} & 0 & s^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q^{-1} & 0 & 0 & 0 & -s^{-1} & 0 & 0 \\ \hline 0 & s & 0 & q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q + q^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q^{-1} & 0 & s^{-1} & 0 \\ \hline 0 & 0 & -s & 0 & 0 & 0 & q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s & 0 & q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \ . \tag{6.14}$$

This Hamiltonian is again proportional to $\hat{R}+I$.
As before, the two-site Hamiltonian $\mathcal{H}^{(j,j+1)}$ can be expressed in terms of the fermionic creation and annihilation operators $c^{\pm}_{\uparrow i}$ and $c^{\pm}_{\downarrow i}$. One finds

$$\begin{aligned}
\mathcal{H}^{(j,j+1)}_{ferm} \;=\; & + s^{-1} c^{+}_{\uparrow j}(1-n_{\downarrow j})c^{-}_{\uparrow j+1}(1-n_{\downarrow j+1}) - s\; c^{-}_{\uparrow j}(1-n_{\downarrow j})c^{+}_{\uparrow j+1}(1-n_{\downarrow j+1}) \\
& + s\; c^{+}_{\downarrow j}(1-n_{\uparrow j})c^{-}_{\downarrow j+1}(1-n_{\uparrow j+1}) - s^{-1} c^{-}_{\downarrow j}(1-n_{\uparrow j})c^{+}_{\downarrow j+1}(1-n_{\uparrow j+1}) \\
& - s^{-1}\sigma^{+}_{j}\sigma^{-}_{j+1} - s\;\sigma^{-}_{j}\sigma^{+}_{j+1} \\
& + q^{-1}(1-n_{\downarrow j})(1-n_{\uparrow j+1}) + q(1-n_{\uparrow j})(1-n_{\downarrow j+1})\;. \qquad (6.15)
\end{aligned}$$

6.3.2 A four parametric Hamiltonian

One can remark that one could have started in Sect. 4 with the following more general four-parameter R-matrix

$$R = \left(\begin{array}{ccc|ccc|ccc}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & q^{-1}q_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -q^{-1}q_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & -q^{-1}\lambda & 0 & q^{-1}q_{12}^{-1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & q^{-2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & q^{-1}q_{23} & 0 & 0 & 0 \\
\hline
0 & 0 & -q^{-1}\lambda & 0 & 0 & 0 & -q^{-1}q_{13}^{-1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -q^{-1}\lambda & 0 & q^{-1}q_{23}^{-1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{array}\right) . \qquad (6.16)$$

Using the fundamental representation and the Casimir operators of the algebra defined by the relations (4.2), with this R-matrix (6.16), one gets the following quantum chain Hamiltonian

$$\mathcal{H} = \left(\begin{array}{ccc|ccc|ccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & q^{-1} & 0 & q_{12}^{-1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & q^{-1} & 0 & 0 & 0 & q_{13}^{-1} & 0 & 0 \\
\hline
0 & q_{12} & 0 & q & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & q+q^{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & q^{-1} & 0 & q_{23}^{-1} & 0 \\
\hline
0 & 0 & q_{13} & 0 & 0 & 0 & q & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & q_{23} & 0 & q & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right) , \qquad (6.17)$$

which reads, in terms of fermionic operators,

$$\begin{aligned}
\mathcal{H}^{(j,j+1)} \;=\; & + q_{12}^{-1} c^{+}_{\uparrow j}(1-n_{\downarrow j})c^{-}_{\uparrow j+1}(1-n_{\downarrow j+1}) - q_{12}\; c^{-}_{\uparrow j}(1-n_{\downarrow j})c^{+}_{\uparrow j+1}(1-n_{\downarrow j+1}) \\
& + q_{23}\; c^{+}_{\downarrow j}(1-n_{\uparrow j})c^{-}_{\downarrow j+1}(1-n_{\uparrow j+1}) - q_{23}^{-1} c^{-}_{\downarrow j}(1-n_{\uparrow j})c^{+}_{\downarrow j+1}(1-n_{\uparrow j+1}) \\
& + q_{13}^{-1}\sigma^{+}_{j}\sigma^{-}_{j+1} + q_{13}\;\sigma^{-}_{j}\sigma^{+}_{j+1} \\
& + q^{-1}(1-n_{\downarrow j})(1-n_{\uparrow j+1}) + q(1-n_{\uparrow j})(1-n_{\downarrow j+1})\;. \qquad (6.18)
\end{aligned}$$

The parameters q_{ij} have the following interpretation: q_{12} (resp. q_{23}) corresponds to a left-right anisotropy of the hopping of the fermionic state $\uparrow$ (resp. $\downarrow$), whereas q_{13} is an anisotropy for the magnetic interaction.
For open boundary conditions, it is easy to find a transformation that eliminates two anisotropy parameters. This transformation is analoguous to the local rescaling that eliminates the anisotropy of magnetic interaction in the XXZ open chain.
The algebra defined by the relations (4.2), with the R-matrix of (6.16), can be brought in the two parameter form of Sect. 4 by rescaling the $\underline{L}^{\pm}$ matrices by the group-like element $Z = q_{12}^{H_1} q_{13}^{-H_1-H_2} q_{23}^{-H_2}$.

6.3.3 $\mathcal{U}_{qs}$-invariant Hamiltonians from the distinguished basis

We now use the distinguished basis and its natural Hopf structure (3.17) (or equivalently the fermionic basis and the alternative coproduct (3.19)) to construct a quantum chain Hamiltonian.

All the Casimir operators again lead to a unique Hamiltonian (up to normalization)

$$\mathcal{H}_{dist} = \left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q & 0 & s^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & 0 & -s^{-1} & 0 & 0 \\ \hline 0 & s & 0 & q^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q+q^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q^{-1} & 0 & s^{-1} & 0 \\ \hline 0 & 0 & -s & 0 & 0 & 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s & 0 & q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) . \tag{6.19}$$

In terms of the fermionic creation and annihilation operators, one has

$$\begin{aligned} \mathcal{H}_{dist}^{(j,j+1)} = {} & + s^{-1} c_{\uparrow j}^{+}(1-n_{\downarrow j}) c_{\uparrow j+1}^{-}(1-n_{\downarrow j+1}) - s\ c_{\uparrow j}^{-}(1-n_{\downarrow j}) c_{\uparrow j+1}^{+}(1-n_{\downarrow j+1}) \\ & + s\ c_{\downarrow j}^{+}(1-n_{\uparrow j}) c_{\downarrow j+1}^{-}(1-n_{\uparrow j+1}) - s^{-1} c_{\downarrow j}^{-}(1-n_{\uparrow j}) c_{\downarrow j+1}^{+}(1-n_{\uparrow j+1}) \\ & - s^{-1}\sigma_j^{+}\sigma_{j+1}^{-} - s\ \sigma_j^{-}\sigma_{j+1}^{+} \\ & + q\ n_{\emptyset j+1} + q^{-1} n_{\emptyset j} + q\ n_{\uparrow j} n_{\downarrow j+1} + q^{-1} n_{\downarrow j} n_{\uparrow j+1}\ . \end{aligned} \tag{6.20}$$

For $s = 1$, this Hamiltonian reduces to the one found in [4], whatever the Casimir operator we use for its construction.

6.3.4 Equivalence with one parameter Hamiltonians

The Hamiltonian (6.15) and the slightly more general Hamiltonian (6.18) are, for open boundary conditions, equivalent to the one parameter Hamiltonian one gets by setting $q_{ij} = 1$ in (6.18). The similarity transformation

$$\mathcal{H} \longrightarrow \mathcal{O}^{-1} \mathcal{H} \mathcal{O} \tag{6.21}$$

with

$$\mathcal{O} = q_{12}^{-\left(\sum_{i<j} n_{\uparrow i} n_{\emptyset j}\right)} q_{23}^{-\left(\sum_{i<j} n_{\emptyset i} n_{\downarrow j}\right)} q_{13}^{-\left(\sum_{i<j} n_{\uparrow i} n_{\downarrow j}\right)} \tag{6.22}$$

indeed brings the Hamiltonian $\mathcal{H}(q, q_{12}, q_{13}, q_{23})$ to the Hamiltonian $\mathcal{H}(q, q'_{12} = 1, q'_{13} = 1, q'_{23} = 1)$. An analogous transformation was also (and earlier) found in [2] in the context of Reaction-Diffusion processes.
The same similarity transformation applied to (6.19) with $q_{12} = -q_{13} = q_{23} = s$ transforms all its non diagonal terms to 1, leading to the Perk–Schultz Hamiltonian $\mathcal{H}^{P,M}$ with $P = 2$, $M = 1$ [12].

Hence, for spin chains with open boundary conditions, the only relevant parameter is q.

After use of this similarity transformation on the Hamiltonians (6.14) and (6.19), they only differ by $q \leftrightarrow q^{-1}$ (reflection of the chain) and the exchange of two diagonal terms. For two or three sites, they are obviously equivalent.

We also proved numerically that their spectra are identical up to seven sites, which strongly suggests that the L-site Hamiltonians are actually equivalent.

From the properties of the $\hat{R}$-matrix (Eqs. (4.7) and (4.8)), we deduce that the translated Hamiltonians $U \equiv \mathcal{H} - q^{\pm 1}$ satisfy the Hecke algebra

$$\begin{aligned} &U_{i,i+1}^2 \pm \lambda U_{i,i+1} - 1 = 0 \, , \\ &U_{i,i+1} U_{i+1,i+2} U_{i,i+1} = U_{i+1,i+2} U_{i,i+1} U_{i+1,i+2} \, , \\ &[U_{i,i+1}, U_{j,j+1}] = 0 \qquad \text{for} \qquad |i-j| \geq 2 \, . \end{aligned} \tag{6.23}$$

6.3.5 Uniqueness of the invariant Hamiltonians

Finally, it is interesting to note that one can compute all the possible $\mathcal{U}_{qs}$-invariant 3^L-state Hamiltonians with nearest neighbour interaction directly, by imposing that

$$[\mathcal{H}_{12}, X_{12}] = 0 \tag{6.24}$$

for all the generators X of $\mathcal{U}_{qs}$. The unique solution up to the identity and a normalization is given by Eq. (6.14) in the fermionic case, and by Eq. (6.19) in the distinguished case. This means that the commutant of $\Delta(\mathcal{U}_{qs})$ in the tensor square of the fundamental representation is generated by I and the matrix $\hat{R}$ (the matrix $\hat{R}$ being either the fermionic or the distinguished one).

The physical consequence is that there are two deformations of the $t - J$ model with $\mathcal{U}_{qs}$-invariance, actually corresponding to the two different Hopf structures (3.4) and (3.19) on $\mathcal{U}_{qs}$.

7 Summary

In the case of classical $sl(1|2)$, we found that the centre is generated by an infinite number of Casimir operators $\mathcal{C}_k$ (k positive integer) of degree k in the Cartan generators. However, the $\mathcal{C}_k$ are not completely independent since they obey quadratic relations. It follows that any two Casimir operators are sufficient to label an irreducible finite dimensional representation of $sl(1|2)$. Using the construction of Sect. 6, one can show that all the Casimir operators $\mathcal{C}_k$ lead to the same two-site Hamiltonian, up to the identity matrix and a normalization factor. Moreover, the same results hold if one uses either the fermionic basis or the distinguished basis of $sl(1|2)$, i.e. the two simple root bases of $sl(1|2)$.

In the deformed case, the situation is more complicated.

In the two-parametric deformed case, the centre of $\mathcal{U}_{qs}$ is generated by an infinite number of Casimir operators $\mathcal{C}_k$ ($k \in \mathbb{Z}$), which obey the same quadratic relations as in the classical case. In the fermionic basis, the Casimir operators $\mathcal{C}_k$ lead to two-site Hamiltonians which are all proportional to the same $\mathcal{H}$.

Such Hamiltonians describe deformed $t - J$ models at the supersymmetric point, the deformation parameter s being interpreted as a left-right anisotropy for the hopping terms and the magnetic interaction and q as an anisotropy in $sl(1|2)$ analoguous to that of the usual XXZ chain. The deformation parameter s (and more generally the parameters q_{ij} of Eq. (6.17)) can be removed, for an open spin chain, by a similarity transformation.

In the distinguished basis, all the Casimir operators lead to a unique Hamiltonian, which is the one found in [4]. It is also a deformation of the $t-J$ model at the supersymmetric point, and for $s = 1$, corresponds to the Perk–Schultz $(1, 2)$ Hamiltonian. We checked numerically that both Hamiltonians (6.14) and (6.19) are actually equivalent.

Lets us finally note that using the conjugate of the fundamental representation amounts to a reversal of the spin chain.

8 Appendix

We give here the explicit relations among the L matrices. In these relations, the value $\zeta = 1$ corresponds to the Z_2-graded case ($\underline{L}$ matrices of Section 4), whereas the value $\zeta = -1$ corresponds to the bosonised case (L matrices of Section 5). For simplicity, we use a single notation L in this Appendix.

$$
\begin{aligned}
&L^{\pm}_{11}L^{+}_{12} = \zeta q^{\mp 1}s^{-1}L^{+}_{12}L^{\pm}_{11} , \\
&L^{\pm}_{22}L^{+}_{12} = q^{\mp 1}s^{-1}L^{+}_{12}L^{\pm}_{22} , \\
&L^{\pm}_{33}L^{+}_{12} = \zeta L^{+}_{12}L^{\pm}_{33} ,
\end{aligned}
\qquad
\begin{aligned}
&L^{\pm}_{11}L^{-}_{21} = \zeta q^{\pm 1}sL^{-}_{21}L^{\pm}_{11} , \\
&L^{\pm}_{22}L^{-}_{21} = q^{\pm 1}sL^{-}_{21}L^{\pm}_{22} , \\
&L^{\pm}_{33}L^{-}_{21} = \zeta L^{-}_{21}L^{\pm}_{33} ,
\end{aligned}
$$

$$
\begin{aligned}
&L^{\pm}_{11}L^{+}_{23} = \zeta L^{+}_{23}L^{\pm}_{11} , \\
&L^{\pm}_{22}L^{+}_{23} = q^{\pm 1}s^{-1}L^{+}_{23}L^{\pm}_{22} , \\
&L^{\pm}_{33}L^{+}_{23} = \zeta q^{\pm 1}s^{-1}L^{+}_{23}L^{\pm}_{33} ,
\end{aligned}
\qquad
\begin{aligned}
&L^{\pm}_{11}L^{-}_{32} = \zeta L^{-}_{32}L^{\pm}_{11} , \\
&L^{\pm}_{22}L^{-}_{32} = q^{\mp 1}sL^{-}_{32}L^{\pm}_{22} , \\
&L^{\pm}_{33}L^{-}_{32} = \zeta q^{\mp 1}sL^{-}_{32}L^{\pm}_{33} ,
\end{aligned}
$$

$$
\begin{aligned}
&L^{\pm}_{11}L^{+}_{13} = q^{\mp 1}s^{-1}L^{+}_{13}L^{\pm}_{11} , \\
&L^{\pm}_{22}L^{+}_{13} = s^{-2}L^{+}_{13}L^{\pm}_{22} , \\
&L^{\pm}_{33}L^{+}_{13} = q^{\pm 1}s^{-1}L^{+}_{13}L^{\pm}_{33} ,
\end{aligned}
\qquad
\begin{aligned}
&L^{\pm}_{11}L^{-}_{31} = q^{\pm 1}sL^{-}_{31}L^{\pm}_{11} , \\
&L^{\pm}_{22}L^{-}_{31} = s^{2}L^{-}_{31}L^{\pm}_{22} , \\
&L^{\pm}_{33}L^{-}_{31} = q^{\mp 1}sL^{-}_{31}L^{\pm}_{33} ,
\end{aligned}
$$

$$
\begin{aligned}
&L^{+}_{12}L^{+}_{12} = 0 , \\
&L^{+}_{23}L^{+}_{23} = 0 ,
\end{aligned}
\qquad
\begin{aligned}
&L^{-}_{21}L^{-}_{21} = 0 , \\
&L^{-}_{32}L^{-}_{32} = 0 ,
\end{aligned}
$$

$$
\begin{aligned}
&L^{+}_{12}L^{+}_{23} + \zeta L^{+}_{23}L^{+}_{12} = \lambda s^{-1}L^{+}_{13}L^{+}_{22} , \\
&L^{+}_{13}L^{+}_{12} = \zeta qsL^{+}_{12}L^{+}_{13} , \\
&L^{+}_{23}L^{+}_{13} = \zeta qs^{-1}L^{+}_{13}L^{+}_{23} ,
\end{aligned}
\qquad
\begin{aligned}
&L^{-}_{32}L^{-}_{21} + \zeta L^{-}_{21}L^{-}_{32} = -\lambda sL^{-}_{31}L^{-}_{22} , \\
&L^{-}_{31}L^{-}_{32} = \zeta q^{-1}s^{-1}L^{-}_{32}L^{-}_{31} , \\
&L^{-}_{31}L^{-}_{21} = \zeta qs^{-1}L^{-}_{21}L^{-}_{31} ,
\end{aligned}
$$

$$
L^{+}_{12}L^{-}_{32} = -q^{-1}sL^{-}_{32}L^{+}_{12} , \qquad L^{+}_{23}L^{-}_{21} = -qsL^{-}_{21}L^{+}_{23} ,
$$

$$
\begin{aligned}
&s^{-1}L^{+}_{12}L^{-}_{31} - \zeta sL^{-}_{31}L^{+}_{12} = \lambda L^{-}_{32}L^{+}_{11} , \\
&s^{-1}L^{+}_{13}L^{-}_{21} - \zeta sL^{-}_{21}L^{+}_{13} = -\zeta\lambda L^{+}_{23}L^{-}_{11} , \\
&s^{-1}L^{+}_{23}L^{-}_{31} - \zeta sL^{-}_{31}L^{+}_{23} = -\lambda L^{-}_{21}L^{+}_{33} , \\
&s^{-1}L^{+}_{13}L^{-}_{32} - \zeta sL^{-}_{32}L^{+}_{13} = \zeta\lambda L^{+}_{12}L^{-}_{33} ,
\end{aligned}
$$

$$
\begin{aligned}
&s^{-1}L^{+}_{12}L^{-}_{21} + \zeta sL^{-}_{21}L^{+}_{12} = -\lambda(L^{+}_{11}L^{-}_{22} - L^{+}_{22}L^{-}_{11}) , \\
&s^{-1}L^{+}_{23}L^{-}_{32} + \zeta sL^{-}_{32}L^{+}_{23} = \zeta\lambda(L^{+}_{22}L^{-}_{33} - L^{+}_{33}L^{-}_{22}) , \\
&s^{-1}L^{+}_{13}L^{-}_{31} - sL^{-}_{31}L^{+}_{13} = \lambda(L^{+}_{11}L^{-}_{33} - L^{+}_{33}L^{-}_{11}) .
\end{aligned}
$$

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