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Translations, Integrals and Fourier Transforms In the Quantum Plane *

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Abstract

We present a formulation of covariant translations in the quantum plane. We are led to an extension of the algebra of the coordinate functions and their dual derivatives by the quantum analogue of their eigenvalues. Jackson exponentials emerge as the corresponding eigenfunctions. An integral invariant under quantum translations is introduced and is used to define quantum Fourier transforms.

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1 Introduction

Since its inception, the quantum plane has been envisioned by many as a paradigm for the general program of q-deformed physics. Such an endeavour presupposes the availability of adequate mathematical tools, integration being one of the most indispensable among them. The aim of this paper is to address aspects of the problem of integration in the quantum plane in a manner that will keep the results accessible to physicists.

We begin, in section 2, by specifying the meaning of a translation in the quantum plane. We find that consistency requires non-trivial commutation relations between the variables that describe translations and the coordinates on the plane as well as the derivatives with respect to these coordinates. We are then able to generate finite translations by q-exponentiating a translation generator. We introduce next, translations for the derivatives and interpret them, together with the ones introduced earlier, as (non-commuting) eigenvalues of the coordinate and derivative operators.

In section 3 we define, along the lines of [1], an integral on the plane essentially by the requirements of linearity and its vanishing when evaluated on a derivative of a function. We are then able to show, using a prescription about how to treat displacements in the integrand, that the above properties imply invariance under finite translations as well.

Section 4 contains a definition of the quantum Fourier transform which is seen to retain, in a q-deformed way, basic properties of its classical analogue. We end the paper by introducing, in section 5, "vacuum projectors". Using them we recover the integration prescription of section 3 in a constructive way.

2 Translations in the q-plane

We recall now the construction of the quantum plane [1] . We deal in the following with the (non-commutative) algebra of functions on the quantum plane enlarged so as to also include derivatives that operate on these functions. We choose as generators of $\mathcal{A}$ the coordinate functions $x^i,\ i = 1, \ldots, n$ (together with the unit function 1_x) and the derivatives dual to them, $\partial_j,\ j = 1, \ldots, n$ (together with the unit 1_∂). A set of consistent commutation relations among the above generators is known:

$$x^i x^j \quad = \quad q^{-1} \hat{R}^{ij}_{kl} x^k x^l$$

$$\begin{aligned}\partial_l \partial_k &= q^{-1} \hat{R}^{ij}_{kl} \partial_j \partial_i \\ \partial_k x^i &= \delta^i_k + q \hat{R}^{ij}_{kl} x^l \partial_j . \end{aligned} \tag{1}$$

Here, $\hat{R}$ is an invertible solution of the quantum Yang-Baxter equation:

$$\hat{R}_{12} \hat{R}_{23} \hat{R}_{12} = \hat{R}_{23} \hat{R}_{12} \hat{R}_{23}$$

and satisfies the characteristic equation:

$$\hat{R}^2 - \lambda \hat{R} - 1 = 0, \qquad \lambda \equiv q - q^{-1} \tag{2}$$

Explicitly, $\hat{R}^{ij}_{kl} = R^{ji}_{kl}$ where R is the $GL_q(n)$ R-matrix of [2], given by:

$$R = q \sum_i e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ii} \otimes e_{jj} + \lambda \sum_{i > j} e_{ij} \otimes e_{ji}$$

where $i, j = 1, \cdots, n$ and e_{ij} is the $n \times n$ matrix with single nonzero element (equal to 1) at (i, j). The above commutation relations permit unambiguous ordering of an arbitrary monomial in the x's and ∂'s into any desired order. One can now write down, if one wishes, differential equations for functions of the x's and study for example quantum mechanical systems by solving Schroedinger's equation in deformed space. In doing so, as well as in many other applications, one is sooner or later bound to be confronted with the problem of (spatially) translating functions of the x's. One place, in particular, where this question would certainly manifest itself, would be in the statement of finite translation invariance of any sort of integral one adopts for the quantum plane. One has then to first make more precise the notion of translation - it is natural, for example, to require a certain covariance. In its simplest form, that would be the requirement that the translated coordinates obey the same algebra as the original ones (we would also need, of course, reduction to the correct classical limit $x^i \mapsto x^i + a^i$ as the deformation parameter approaches its classical value). In bialgebra language, we are in search of a coproduct for the algebra of the x's, namely a map $\Delta : \mathcal{A}_x \to \mathcal{A}_x \otimes \mathcal{A}_x$ that is an algebra homomorphism: $\Delta(xy) = \Delta(x)\Delta(y), x, y \in \mathcal{A}_x$, and satisfies some standard additional conditions (see for example [3] - here $\mathcal{A}_x$ is the subalgebra of $\mathcal{A}$ generated by the coordinates). Such a map has been sought for for quite some time now, without success. Faced with this fact, we elect to pursue a different approach. We introduce a set of "displacements" a^i, $i = 1, \ldots, n$ and require that the $x + a$'s obey the same commutation relations as the x's. We would also like the displacements to be of "coordinate nature" *i.e.* we postulate $a - a$ commutation relations identical to those of the x's. We are then forced to introduce non-trivial $a - x$ commutation relations. The resulting algebra is:

$$\begin{aligned} a^i a^j &= q^{-1} \hat{R}^{ij}_{kl} a^k a^l \\ x^i a^j &= q \hat{R}^{ij}_{kl} a^k x^l. \end{aligned} \tag{3}$$

We check for covariance:

$$\begin{aligned} (x_1 + a_1)(x_2 + a_2) &= x_1 x_2 + a_1 x_2 + x_1 a_2 + a_1 a_2 \\ &= x_1 x_2 + (q^{-1} \hat{R}^{-1}_{12} + 1) x_1 a_2 + a_1 a_2. \end{aligned}$$

Also:

$$\begin{aligned} q^{-1} \hat{R}_{12} (x_1 + a_1)(x_2 + a_2) &= q^{-1} \hat{R}_{12} x_1 x_2 + q^{-2} x_1 a_2 \\ &\quad + q^{-1} \hat{R}_{12} x_1 a_2 + q^{-1} \hat{R}_{12} a_1 a_2. \end{aligned} \tag{4}$$

Using now the characteristic equation (2) in the form:

$$q^{-1} \hat{R}^{-1} + 1 = q^{-1} \hat{R} + q^{-2}$$

and comparing the two expressions above we find:

$$(x_1 + a_1)(x_2 + a_2) = q^{-1} \hat{R}_{12} (x_1 + a_1)(x_2 + a_2)$$

as desired. It is interesting to compare (3) with the commutation relations between coordinates and differentials introduced in [1] :

$$x_1 \xi_2 = q \hat{R}_{12} \xi_1 x_2 \ ;$$

the displacements a^i are the bosonic analogue of the ξ's (the algebra (3) has been introduced by Majid, in the context of braided Hopf algebras, in [4]). We can also give consistent $\partial - a$ commutation relations:

$$\partial_k a^i = q^{-1} (\hat{R}^{-1})^{ij}_{kl} a^l \partial_j \tag{5}$$

(again, similar to the $\partial - \xi$ ones). Consider now the translation generator T defined by:

$$T \equiv a^i \partial_i \equiv a \cdot \partial.$$

Using (3), (5), we easily find:

$$[T, x^i] = a^i, \quad T \partial_i = q^2 \partial_i T, \quad T a^i = q^{-2} a^i T. \tag{6}$$

These allow us to build a finite translation operator by "q-exponentiation". We have:

$$T^n x^i = x^i T^n + [n]_q T^{n-1} a^i \tag{7}$$

where $[n]_q \equiv (1-q^{2n})/(1-q^2)$ and therefore:

$$x^i e_q(T) = e_q(T)(x^i - a^i) \tag{8}$$

where:

$$e_q(T) \equiv \sum_{n=0}^{\infty} \frac{1}{[n]_q!} T^n, \quad [n]_q! \equiv [1]_q [2]_q \ldots [n]_q$$

is the Jackson exponential (see for example [5] and references therein). Alternatively, we can write (7) in the form:

$$T^n x^i = x^i T^n + [n]_{q^{-1}} a^i T^{n-1}$$

which gives:

$$e_{q^{-1}}(T) x^i = (x^i + a^i) e_{q^{-1}}(T) \tag{9}$$

or, more generally:

$$e_{q^{-1}}(T) f(x) = f(x+a) e_{q^{-1}}(T). \tag{10}$$

One can regard (8) (or (9)) as an eigenvalue equation for the operator x^i. To make this more precise, we introduce coordinate and derivative vacua, denoted by $|\Omega_x\rangle$ and $|\Omega_\partial\rangle$ respectively, which satisfy:

$$x^i |\Omega_x\rangle = 0, \quad 1_x |\Omega_x\rangle = |\Omega_x\rangle, \quad \partial_i |\Omega_\partial\rangle = 0, \quad 1_\partial |\Omega_\partial\rangle = |\Omega_\partial\rangle$$

with similar relations for x, ∂ acting from the right:

$$\langle\Omega_x| x^i = 0, \quad \langle\Omega_x| 1_x = \langle\Omega_x|, \quad \langle\Omega_\partial| \partial_i = 0, \quad \langle\Omega_\partial| 1_\partial = \langle\Omega_\partial|.$$

The action of x^i on a function $f(\partial, a)$, denoted by $x^i(f(\partial, a))$, is expressed in terms of the coordinate vacuum as:

$$x^i(f(\partial, a))|\Omega_x\rangle = x^i f(\partial, a)|\Omega_x\rangle. \tag{11}$$

In words, to compute the left hand side of (11), we order it with all the x's on the right, where they anihilate the vacuum, and what remains is termed "the action of x^i on $f(\partial, a)$". We can define the (more familiar) action of the derivatives on functions of x, a in a similar manner. Actions from the right are also obviously defined via "left vacua" $\langle\Omega_x|$, $\langle\Omega_\partial|$. With these (standard) definitions, (8) gives:

$$(e_q(T)) x^i = e_q(T) a^i \tag{12}$$

which suggests the interpretation of a^i as the eigenvalue of x^i (x^i acts here from the right). In the classical limit $q \to 1$, the a's commute with everything. Notice that $e_q(T)$ is a common eigenfunction for all x^i, the noncommutativity of the latter being reflected in the non-trivial $a - a$ commutation relations. It is interesting to note that one can interpret the derivatives ∂_i the way one does in classical analysis: $\partial_i(f(x))$ is the coefficient of a^i in the expansion of $f(x+a)$ around x [6]. Indeed, from (10) we have:

$$e_{q^{-1}}(T)(f(x)) = f(x+a) \tag{13}$$

which, by expanding the Jackson exponential, gives:

$$f(x+a) = f(x) + a^i \partial_i(f(x)) + \mathcal{O}(a^2), \tag{14}$$

the only difference in the quantum case being that one has to specify an ordering before identifying the derivative (above, we took the a's to stand to the left of the x's).

The interpretation given to the a^i above naturally leads to the question whether a similar construction is possible for the derivatives. To this end, we introduce the momentum-space analogue of the a's, which we call $p_i, \quad i = 1, \ldots, n$ and find that the following commutation relations are consistent with the rest of the algebra:

$$\begin{aligned} p_l p_k &= q^{-1} \hat{R}^{ij}_{kl} p_j p_i \\ p_l \partial_k &= q \hat{R}^{ij}_{kl} \partial_j p_i \\ p_k x^i &= q^{-1} (\hat{R}^{-1})^{ij}_{kl} x^l p_j \\ p_k a^i &= q^{-1} (\hat{R}^{-1})^{ij}_{kl} a^l p_j . \end{aligned} \tag{15}$$

A host of useful identities can now be computed. We give a list involving invariant bilinears ($\alpha \cdot \beta \equiv \alpha^i \beta_i$):

$$\begin{aligned} T p_i - p_i T &= 0 \\ x \cdot \partial\, x^i &= x^i + q^2 x^i\, x \cdot \partial \\ x \cdot \partial\, a^i - a^i\, x \cdot \partial &= 0 \\ \partial_i\, x \cdot \partial &= \partial_i + q^2 x \cdot \partial\, \partial_i \\ x \cdot \partial\, p_i - p_i\, x \cdot \partial &= 0 \\ x \cdot p\, x^i &= q^{-2} x^i\, x \cdot p \\ x \cdot p\, a^i - a^i\, x \cdot p &= 0 \\ \partial_i\, x \cdot p &= x \cdot p\, \partial_i + p_i \end{aligned}$$

$$
\begin{aligned}
x \cdot p\, p_i &= q^2 p_i\, x \cdot p \\
a \cdot p\, x^i &= q^{-2} x^i\, a \cdot p - q^{-1}\lambda a^i\, x \cdot p \\
a \cdot p\, a^i &= q^{-2} a^i\, a \cdot p \\
\partial_i\, a \cdot p &= q^{-2} a \cdot p\, \partial_i - q^{-1}\lambda T p_i \\
p_i\, a \cdot p &= q^{-2} a \cdot p\, p_i \\
(1 + q\lambda x \cdot \partial)\, x^i &= q^2 x^i (1 + q\lambda x \cdot \partial) \\
\partial_i (1 + q\lambda x \cdot \partial) &= q^2 (1 + q\lambda x \cdot \partial)\, \partial_i \\
(x \cdot p)(a \cdot p) &= q^2 (a \cdot p)(x \cdot p).
\end{aligned} \tag{16}
$$

We easily find now how ∂_i commutes with $e_q(x \cdot p)$:

$$\partial_i e_q(x \cdot p) = e_q(x \cdot p)\partial_i + p_i e_q(x \cdot p). \tag{17}$$

Also:

$$\partial_i e_{q^{-1}}(x \cdot p) = e_{q^{-1}}(x \cdot p)\partial_i + e_{q^{-1}}(x \cdot p) p_i. \tag{18}$$

We can therefore interpret the p's as (non-commuting) eigenvalues of the derivatives:

$$\partial_i (e_q(x \cdot p)) = p_i e_q(x \cdot p)\,, \tag{19}$$

Notice that being eigenvalues of derivatives, rather than momenta, the p's become real commuting quantities in the classical limit.

A couple of remarks are in order here. The first regards the covariance of the scheme described above under the coaction of $GL_q(n)$. The commutation relations given in (1), (3), (5) and (15) go into themselves when x, ∂, a and p transform according to:

$$
\begin{aligned}
x^i \mapsto (x')^i &= T^i_j x^j \\
a^i \mapsto (a')^i &= T^i_j a^j \\
\partial_i \mapsto (\partial')_i &= \partial_j M^j_i \\
p_i \mapsto (p')_i &= \partial_j M^j_i
\end{aligned}
$$

where T^i_j is a $GL_q(n)$ matrix, $M^t = (T^t)^{-1}$ (M^t denotes the transpose of M) and we take, as in [1], the elements of T to commute with all the variables and derivatives above. A second point that deserves attention is the fact that derivations do not commute with translations. In general:

$$\partial_i (f(x + a)) \neq \partial_i (f(x))|_{x \mapsto x+a}. \tag{20}$$

This can be traced to the fact that ∂_i does not commute with $a \cdot \partial$. Indeed, in order for (20) to be an equality, we would need (using (14)):

$$\begin{aligned} \partial_i(f(x) + a \cdot \partial(f(x))) &= \partial_i(f(x)) + a \cdot \partial(\partial_i(f(x))) \Rightarrow \\ \Rightarrow \partial_i(a \cdot \partial(f(x))) &= a \cdot \partial(\partial_i(f(x))) \Rightarrow \\ \Rightarrow \partial_i\, a \cdot \partial &= a \cdot \partial\, \partial_i \end{aligned}$$

while, in our case, (6) holds: $a \cdot \partial\, \partial_i = q^2 \partial_i\, a \cdot \partial$ (one can make a different choice of commutation relations which will make (20) into an equality but then (14) is not satisfied - we will not explore this further here).

3 Invariant Integration

We wish now to turn our attention to the problem of integration. The integral we are looking for is a linear map from functions on the quantum plane to complex numbers. Keeping in mind the classical limit, we expect it to be defined only for a class $\mathcal{A}_x^I$ of elements of $\mathcal{A}_x$ - we will use the notation $\langle f \rangle$ for the average, or integral, of f in that class. Such a map acquires interest when endowed with specific covariance properies. In our case, it is natural to require invariance under translations. This can be expressed in infinitesimal form as the requirement that the integral of a derivative vanish:

$$\langle\, \partial_i(f(x))\, \rangle = 0, \quad f \in \mathcal{A}_x^I. \tag{21}$$

A prescription for computing such an integral is known [1] . One first expands $f(x)$ in a sum of monomials in the x's and uses the commutation relations to bring each such monomial into some standard ordering (the same for all monomials). Then one performs the classical integral (from minus infinity to plus infinity) of the ordered function - the result is the quantum integral $\langle f \rangle$. Different standard orderings of the x's change only the overall normalization and the result satisfies (21)(notice that ∂_i is the *quantum* derivative). We would like though to be able to talk about finite translation invariance, *i.e.* we would like our integral to satisfy an equation like

$$\langle f(x + a) \rangle = \langle f(x) \rangle. \tag{22}$$

To make this precise, we ought to generalize the prescription for integration given above to the case of a function of x *and* a (since x, a do not commute, such a generalization is not trivial). Nevertheless, the natural approach works: to compute $\langle f(x + a) \rangle$, expand in monomials of x, a, use the commutation relations to move all the a's, say, to the left of each monomial and out of the integral, and then compute the quantum integral of the x's as before (notice that the a's need not be brought into any standard order). That such an integral satisfies (22) can easily be seen as follows. From (13) we have:

$$\begin{aligned}\langle f(x+a)\rangle &= \langle e_{q^{-1}}(a\cdot\partial)(f(x))\rangle \\ &= \sum_{n=0}^{\infty}\frac{1}{[n]_{q^{-1}}!}\langle (a\cdot\partial)^n(f(x))\ .\end{aligned}$$

We may now use the commutation relations given in (5) to move all the a's in $(a\cdot\partial)^n$ to the left (and then out of the integral). The form of (5) ensures that each term left in the integrand, except for $n=0$, will be the derivative of some function of the x's and the integral of these vanishes by (21); (22) then follows. We should emphasize here that the integral of $f(x,a)$ is not, in general, translationally invariant (*i.e.* while $\langle f(x+a)\rangle = \langle f(x)\rangle$ holds, $\langle f(x+a,a)\rangle \neq \langle f(x,a)\rangle$ in general). In the same spirit, we define the integral $\langle f(x,a,p)\rangle$: we move a and p to the left and then perform quantum integration on the x's - we 'll need this in defining the Fourier transform in the next section. We close this section with a formula for integration by parts, valid when the Jackson exponential of $x\cdot p$ is in the integrand. From (17) we get:

$$\partial_i(e_q(x\cdot p)f(x)) = e_q(x\cdot p)\partial_i(f(x)) + p_i e_q(x\cdot p)f(x)$$

which, upon integration, gives:

$$\begin{aligned}\langle e_q(x\cdot p)\partial_i(f(x))\rangle &= -p_i\langle e_q(x\cdot p)f(x)\rangle \\ &= -\langle \partial_i(e_q(x\cdot p))f(x)\rangle \qquad (23)\end{aligned}$$

4 Fourier Transform

Armed with the tools developed in the previous section, we are now (almost) ready to discuss Fourier transforms in the quantum plane. The only ingredient missing is the observation that

$$\langle f(x+a,p)\rangle = \langle f(x,p)\rangle. \qquad (24)$$

To see this, write $f(x,p)$ in the form:

$$f(x,p) = \sum_i g_i(p)f_i(x)$$

to get:

$$\begin{aligned} e_{q^{-1}}(T)(f(x,p)) &= \sum_i e_{q^{-1}}(T)(g_i(p)f_i(x)) \\ &= \sum_i g_i(p)e_{q^{-1}}(T)(f_i(x)) \\ &= \sum_i g_i(p)f_i(x+a) \end{aligned} \tag{25}$$

where, in the second line, we used the fact that p commutes with T. A look at (15) shows that the commutation relations of $p, x+a$ are identical to those of p, x which allows us to sum the last line of (25):

$$e_{q^{-1}}(T)(f(x,p)) = f(x+a,p). \tag{26}$$

Integrating both sides of (25) and using (26) we then get:

$$\begin{aligned} \langle f(x+a,p)\rangle &= \sum_i g_i(p)\langle f_i(x+a)\rangle \\ &= \sum_i g_i(p)\langle f_i(x)\rangle \\ &= \langle f(x,p)\rangle. \end{aligned}$$

We define now the Fourier transform $\tilde{f}(p)$ of a function $f(x)$ by:

$$\tilde{f}(p) \equiv \langle e_q(-ix\cdot p)f(x)\rangle \ . \tag{27}$$

We will need the following properties of the Jackson exponential

$$e_q(\alpha+\beta) = e_q(\beta)e_q(\alpha) \text{ for } \alpha\beta = q^2\beta\alpha \tag{28}$$

and

$$e_q(\alpha)e_{q^{-1}}(-\alpha) = 1 \tag{29}$$

to derive the analogue of a property of Fourier transforms, familiar in the classical case. Setting $f(x+a) \equiv f_a(x)$ we have:

$$\begin{aligned} \tilde{f}(p) &= \langle e_q(-ix\cdot p)f(x)\rangle \\ &= \langle e_q(-i(x+a)\cdot p)f_a(x)\rangle \\ &= \langle e_q(-ia\cdot p)e_q(-ix\cdot p)f_a(x)\rangle \\ &= e_q(-ia\cdot p)\tilde{f}_a(p) \Rightarrow \\ \Rightarrow \tilde{f}_a(p) &= e_{q^{-1}}(ia\cdot p)\tilde{f}(p) \end{aligned} \tag{30}$$

(where in the third line we used the last of (16)). Notice that, as in the classical case, the factor in front of $\tilde{f}(p)$ is actually a one dimensional representation of translations. Under $x \mapsto x+a$:

$$\begin{aligned} e_{q^{-1}}(ix\cdot p) &\mapsto e_{q^{-1}}(ix\cdot p + ia\cdot p) \\ &= e_{q^{-1}}(ix\cdot p)e_{q^{-1}}(ia\cdot p). \end{aligned}$$

5 Vacuum Projectors

In this section we introduce a "vacuum projector" E which realizes the operator $|\Omega_\partial\rangle\langle\Omega_x|$ (up to a possible normalization factor) in terms of coordinates and derivatives (a similar object, in a Hopf algebra context, has been introduced in [7]). It is given by the formal expansion:

$$E = \sum_{k=0}^{\infty} \frac{(-1)^k}{[k]_{q^{-1}}!} x^{i_1} \ldots x^{i_k} \partial_{i_k} \ldots \partial_{i_1}. \tag{31}$$

We scetch the proof. Setting:

$$E_k \equiv x^{i_1} \ldots x^{i_k} \partial_{i_k} \ldots \partial_{i_1}$$

we find:

$$E_k x^i = [k]_q x^i E_{k-1} + q^{2k} x^i E_k$$

(x^i is the i-th coordinate). We then have:

$$\begin{aligned} E x^i &= \sum_{k=0}^{\infty} \frac{(-1)^k}{[k]_{q^{-1}}!} E_k x^i \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k}{[k]_q!} q^{k(k-1)} [k]_q x^i E_{k-1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{[k]_q!} q^{k(k-1)} q^{2k} x^i E_k \\ &= -\sum_{k=0}^{\infty} \frac{(-1)^k}{[k]_q!} q^{k(k+1)} x^i E_k + \sum_{k=0}^{\infty} \frac{(-1)^k}{[k]_q!} q^{k(k+1)} x^i E_k \\ &= 0. \end{aligned}$$

As a result, $E^2 = E$. One similarly checks that $\partial_i E = 0$.

We can now easily realize the projector $|\Omega_x\rangle\langle\Omega_\partial|$ as well. We know from [1] that $\mathcal{A}$ admits the $*$-involution (which we denote by a bar):

$$\bar{x^i} = x^i, \qquad \bar{\partial}_i = -q^{2(n+1-i)} \partial_i, \qquad \bar{q} = q^{-1}$$

(corresponding to a real quantum plane). It then follows immediately that $\bar{E}$, given explicitly by:

$$\bar{E} = \sum_{k=0}^{\infty} \frac{1}{[k]_{q^{-1}}!} q^{2k(n+1)} q^{-2(i_1+i_2+\ldots+i_k)} \bar{E}_k \tag{32}$$

where $\bar{E}_k \equiv \partial_{i_1} \ldots \partial_{i_k} x^{i_k} \ldots x^{i_1}$, realizes the operator $|\Omega_x\rangle\langle\Omega_\partial|$. An alternative form for E_k, as a function of $x \cdot \partial$, is:

$$E_k = q^{-k(k-1)} (x \cdot \partial)(x \cdot \partial - [1]_q)(x \cdot \partial - [2]_q) \ldots (x \cdot \partial - [n-1]_q).$$

One can easily show that $\bar{E}_k$ can also be expressed in terms of $x \cdot \partial$.

The above objects allow us to approach the problem of integration from an alternative point of view. We can use the vacua introduced earlier to define the integral of a function $f(x)$ via:

$$\langle f(x) \rangle = \langle \Omega_\partial | f(x) | \Omega_\partial \rangle. \tag{33}$$

This definition automatically satisfies $\langle \partial_i (f(x)) \rangle = 0$ and therefore it also satisfies $\langle f(x+a) \rangle = \langle f(x) \rangle$. Notice however that in deriving this last invariance property we do not need any ad-hoc rules about how to commute a's (or, for that matter, p's) through the "integral sign". Indeed, choosing the normalization $E = |\Omega_\partial\rangle\langle\Omega_x|$, $\bar{E} = |\Omega_x\rangle\langle\Omega_\partial|$ (which, in turn, implies $\langle\Omega_x|\Omega_\partial\rangle = 1$) (33) gives:

$$\begin{aligned} \bar{E} f(x) E &= |\Omega_x\rangle\langle\Omega_\partial|f(x)|\Omega_\partial\rangle\langle\Omega_x| \\ &= \langle f(x)\rangle |\Omega_x\rangle\langle\Omega_x| \\ &\equiv \langle f(x)\rangle \delta(x). \end{aligned}$$

However, as we have seen, E_k and $\bar{E}_k$ can be expressed as functions of $x \cdot \partial$ only. Refering back to the list given in (16), we see that a^i, p^j commute with $x \cdot \partial$ and this justifies postulating the integration procedure described in section 3.

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It is a pleasure to dedicate this paper to Abdus Salam.

Quantum plane commutation relations, $n = 2$

We give here explicitly, the full set of commutation relations for the two-dimensional case. The $\hat{R}$-matrix in this case is:

$$\hat{R} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \tag{34}$$

from which we get, in obvious notation:

$$\begin{array}{lrcl}
\underline{x-x} & xy & = & qyx \\
 & & & \\
\underline{x-\partial} & \partial_x x & = & 1 + q^2 x\partial_x + q\lambda y\partial_y \\
 & \partial_x y & = & qy\partial_x \\
 & \partial_y x & = & qx\partial_y \\
 & \partial_y y & = & 1 + q^2 y\partial_y \\
\underline{x-a} & xa & = & q^2 ax \\
 & xb & = & qbx + q\lambda ay \\
 & ya & = & qay \\
 & yb & = & q^2 by \\
\underline{x-p} & p_x x & = & q^{-2} xp_x \\
 & p_x y & = & q^{-1} yp_x \\
 & p_y x & = & q^{-1} xp_y \\
 & p_y y & = & q^{-2} yp_y - q^{-1}\lambda xp_x \\
 & & & \\
\underline{\partial-\partial} & \partial_x\partial_y & = & q^{-1}\partial_y\partial_x \\
 & & & \\
\underline{\partial-a} & \partial_x a & = & q^{-2} a\partial_x \\
 & \partial_x b & = & q^{-1} b\partial_x \\
 & \partial_y a & = & q^{-1} a\partial_y \\
 & \partial_y b & = & q^{-2} b\partial_y - q^{-1}\lambda a\partial_x
\end{array} \tag{35}$$

$$
\begin{array}{lrcl}
\underline{\partial - p} & p_x\partial_x & = & q^2\partial_x p_x \\
 & p_x\partial_y & = & q\partial_y p_x \\
 & p_y\partial_x & = & q\partial_x p_y + q\lambda\partial_y p_x \\
 & p_y\partial_y & = & q^2\partial_y p_y \\
 & & & \\
\underline{a - a} & ab & = & qba \\
 & & & \\
\underline{a - p} & p_x a & = & q^{-2}ap_x \\
 & p_x b & = & q^{-1}bp_x \\
 & p_y a & = & q^{-1}ap_y \\
 & p_y b & = & q^{-2}bp_y - q^{-1}\lambda ap_x \\
 & & & \\
\underline{p - p} & p_x p_y & = & q^{-1}p_y p_x
\end{array}
\qquad (36)
$$

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