(5) To prove that $\tilde{M}_{\prec} \cap \mathrm{Y}(\widehat{\mathrm{sl}}(2))$ constitute a linearly independent set, we use representations $\rho_{z_1,\ldots,z_n} \otimes \pi_2^{\otimes j}$ as in the third step.

(6) Thus, we have proved the PBW theorem for $\mathrm{Y}(\widehat{\mathrm{sl}}(2))$; the PBW theorem for $\mathrm{Y}(\widetilde{\mathrm{sl}}(2))$ is an easy corollary.

Thus, the PBW theorem is proved.

Denote by $\mathrm{Y}^{\pm} = \mathrm{Y}^{\pm}(\widehat{\mathrm{sl}}(2))$ the subalgebra generated by $h_{ik}, x_{jl}^{+}, x_{rs}^{-}$ with non-negative i, j, r, and put

$$\mathrm{Y}^{\pm\pm} = \mathrm{Y}^{\pm\pm}(\widehat{\mathrm{sl}}(2)) = \text{linear span}\{X \in M_{\prec} \cap \mathrm{Y}^{\pm} \mid \pm \deg_1 X > 0\}.$$

In each of the relations (I)–(VI), the $\deg_1$'s of all the terms are the same, so $\mathrm{Y}^{\pm}(\widehat{\mathrm{sl}}(2))$ and $\mathrm{Y}^{\pm\pm}(\widehat{\mathrm{sl}}(2))$ are subalgebras. One easily sees that $M_{\prec} \cap \mathrm{Y}^{\pm}$ is a PBW basis in $\mathrm{Y}^{\pm}$, and $M_{\prec} \cap \mathrm{Y}^{\pm\pm}$ -in $\mathrm{Y}^{\pm\pm}$, and the following isomorphisms are valid:

$$\mathrm{Y}^{+} \cong \mathrm{Y}^{0} \otimes \mathrm{Y}^{++}, \quad \mathrm{Y}^{-} \cong \mathrm{Y}^{--} \otimes \mathrm{Y}^{0},$$

$$\mathrm{Y}(\widehat{\mathrm{sl}}(2)) \cong \mathrm{Y}^{--} \otimes \mathrm{Y}^{0} \otimes \mathrm{Y}^{++} \otimes \mathbb{C}[c],$$

$$\mathrm{Y}(\widetilde{\mathrm{sl}}(2)) \cong \mathrm{Y}^{--} \otimes \mathrm{Y}^{0} \otimes \mathrm{Y}^{++} \otimes \mathbb{C}[c] \otimes \mathbb{C}[d].$$

Now we describe some morphisms of the algebra and coalgebra structures of $\mathrm{Y}(\widetilde{\mathrm{sl}}(2))$, $\mathrm{Y}(\widehat{\mathrm{sl}}(2))$, and of $\mathrm{Y}(\mathrm{sl}(2) \otimes \mathbb{C}[t^{-1}, t]) := \mathrm{Y}(\widehat{\mathrm{sl}}(2))/\{c = 0\}$.

The first one is an automorphism (for both the algebra and coalgebra structures) defined by

$$x_k^{+}(u)^{\pm} \mapsto x_k^{-}(u)^{\mp}, \qquad x_k^{-}(u)^{\pm} \mapsto x_k^{+}(u)^{\mp}, \qquad h_1(u)^{\pm} \mapsto h_1(u)^{\mp}.$$

The second is a two-parameter family of automorphisms $T_{\lambda,a}$ of $\mathrm{Y}(\mathrm{sl}(2) \otimes C[t^{-1}, t])$, $\lambda \neq 0$, $a \in \mathbb{C}$ defined by $d \mapsto d$, $x_{k0}^{\pm} \mapsto \lambda^{k} x_{k0}^{\pm}$, $h_{01} \mapsto h_{01} + a h_{00}$; for $\mathrm{Y}(\widehat{\mathrm{sl}}(2))$ (and $\mathrm{Y}(\widetilde{\mathrm{sl}}(2))$), we have only a one-parameter family

$$c \mapsto c, \qquad (d \mapsto d), \qquad x_{k0}^{\pm} \mapsto x_{k0}^{\pm}, \qquad h_{01} \mapsto h_{01} + a h_{00}.$$

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Complex Quantum Enveloping Algebras as Twisted Tensor Products

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Abstract. We introduce a $*$-structure on the quantum double and its dual in order to make contact with various approaches to the enveloping algebras of complex quantum groups. Furthermore, we introduce a canonical basis in the quantum double, its universal R-matrices and give its relation to subgroups in the dual Hopf algebra.

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1. Quantum Double as Quasitriangular Twisted Tensor Product Hopf Algebra

During recent years, the q-deformation of the Lorentz group and the Lorentz algebra have gained a lot of interest [1–3, 6, 9–11]. In order to clarify the relation between different descriptions, in the following we investigate an alternative approach to complex quantum groups and their universal enveloping algebras as defined in [5]. For the quantum group, it corresponds to the construction in [10]. The most simple example of these enveloping algebras of complex quantum groups is the Lorentz algebra. The starting point of our investigation is the quantum double in the form introduced in [12]. We introduce a $*$-operation on the quantum double and construct a new canonical system of generators which can be identified with the defining generators in [3, 5]. This identification then allows a straightforward reduction of generators for all quantum enveloping algebras as defined in [5]. On the real representation of the complex quantum group, we give an explicit description for the subgroup homomorphism induced by the embedding of the Hopf algebra $\mathcal{U}$ associated to A_{n-1}, B_n, C_n and D_n in its quantum double.

Let $(\mathcal{U}, \mathcal{R})$ and $(\hat{\mathcal{U}}, \hat{\mathcal{R}})$ be two quasi-triangular Hopf algebras (QTHAs). Denote by $\bar{\mathcal{U}}$ the tensor product of $\mathcal{U}$ and $\hat{\mathcal{U}}$ as $\mathbb{C}$ vector spaces.

$$\bar{\mathcal{U}} = \mathcal{U} \otimes \hat{\mathcal{U}}. \tag{1}$$

Then the following definitions

$$
\begin{aligned}
&\bar{m} = (m \otimes \dot{m}) \circ \tau_{\dot{1}2}, \qquad \bar{\eta}(\lambda) = \lambda\bar{1}, \qquad \bar{1} = 1 \otimes \dot{1},\\
&\bar{\Delta} = \tau_{\dot{1}2} \circ (\Delta \otimes \dot{\Delta}), \qquad \bar{\varepsilon} = \varepsilon \otimes \dot{\varepsilon}, \\
&\bar{S} = S \otimes \dot{S}, \qquad \bar{\mathscr{R}} = \bar{\mathscr{R}}_{1\dot{1}2\dot{2}} = \mathscr{R}_{12}\dot{\mathscr{R}}_{\dot{1}\dot{2}}
\end{aligned}
\tag{2}
$$

render $(\bar{\mathscr{U}}, \bar{\mathscr{R}})$ a new QTHA, the tensor product QTHA. For simplicity, we will refer to this as the tensor Hopf algebra. Since, with $(\mathscr{U}, \mathscr{R})$ also $(\mathscr{U}, \mathscr{R}_{21}^{-1})$ is a QTHA, we can choose

$$(\dot{\mathscr{U}}, \dot{\mathscr{R}}_{\dot{1}\dot{2}}) = (\mathscr{U}, \mathscr{R}_{21}^{-1}) \tag{3}$$

the dotted mappings coinciding with the undotted, yielding the tensor Hopf algebra

$$(\bar{\mathscr{U}}, \bar{\mathscr{R}}) = (\mathscr{U} \otimes \mathscr{U}, \mathscr{R}_{12}\mathscr{R}_{\dot{2}\dot{1}}^{-1}). \tag{4}$$

$\dot{\mathscr{R}} = \mathscr{R}$ would do as well but $\mathscr{R}_{21}^{-1}$ leads to an $\boldsymbol{\mathscr{R}}$ below which is symmetric in $\mathscr{R}$ and $\mathscr{R}^{-1}$ and as a consequence has the property $\boldsymbol{\mathscr{R}}^* = \boldsymbol{\mathscr{R}}^{-1}$ (cf. Section 3). The canonical construction of the QUEA and its dual based on the fundamental representation of either of those choices leads to the same Hopf algebras differing only in the quasi-triangular structure of the former.

THEOREM 1. *With*

$$\boldsymbol{\mathscr{U}} = \bar{\mathscr{U}} \text{ as vector space over } \mathbb{C}, \tag{5}$$

$$\boldsymbol{m} = \bar{m}, \tag{6}$$

$$\boldsymbol{\Delta} = \mathscr{R}_{\dot{1}2}^{-1}\bar{\Delta}\mathscr{R}_{\dot{1}2}, \tag{7}$$

$$\boldsymbol{\varepsilon} = \bar{\varepsilon}, \tag{8}$$

$$\boldsymbol{S} = \mathscr{R}_{\dot{1}1}\bar{S}\mathscr{R}_{\dot{1}1}^{-1}, \tag{9}$$

$$\boldsymbol{\mathscr{R}} = \mathscr{R}_{\dot{2}1}^{-1}\bar{\mathscr{R}}\mathscr{R}_{\dot{1}2} = \mathscr{R}_{\dot{2}1}^{-1}\mathscr{R}_{\dot{2}\dot{1}}^{-1}\mathscr{R}_{12}\mathscr{R}_{\dot{1}2}, \tag{10}$$

$(\boldsymbol{\mathscr{U}}, \boldsymbol{\mathscr{R}})$ *becomes a QTHA.*

In [12], *this theorem and the coincidence of this construction with Drinfeld's quantum double have been proved.*

$(\boldsymbol{\mathscr{U}}, \boldsymbol{\mathscr{R}})$ can be interpreted as the result of twisting $(\bar{\mathscr{U}}, \bar{\mathscr{R}})$ (in the sense of Drinfeld [4]) with

$$\bar{\mathscr{F}} = \bar{\mathscr{F}}_{1\dot{1}2\dot{2}} = \mathscr{R}_{\dot{1}2}^{-1} \tag{11}$$

giving

$$\bar{v} = \bar{\mathscr{F}}^{(1)}\bar{S}(\bar{\mathscr{F}}^{(2)}) = (1 \otimes \mathscr{R}^{-1(1)})(S(\mathscr{R}^{-1(2)}) \otimes 1) = \mathscr{R}_{\dot{1}1}. \tag{12}$$

Introducing the dual Hopf algebra of $\mathscr{U}$ – in the sense of Majid [7] – denoted by $\mathscr{A}$, the matrix of whose generators u^i_j constitute the fundamental representation of $\mathscr{U}$

and utilizing the expressions for the RTF-generators $\ell^{\pm i}_{j}$ [7, 13] as fundamental semi-representations of $\mathscr{R}$ and $\mathscr{R}^{-1}$

$$\ell^{+} = \langle \cdot \otimes u, \mathscr{R} \rangle, \qquad \ell^{-} = \langle u \otimes \cdot, \mathscr{R}^{-1} \rangle, \tag{13}$$

we obtain a complete set of generators of $\boldsymbol{\mathscr{U}}$ as well as of $\bar{\mathscr{U}}$:

$$\ell^{\pm} \otimes 1, \qquad 1 \otimes \ell^{\pm} \tag{14}$$

A new set of generators of $\boldsymbol{\mathscr{U}}$ is obtained by the following definitions:*

$$\ell_{\bar{1}}^{++} = \langle \cdot \otimes : \otimes u_1 \otimes 1, \boldsymbol{\mathscr{R}} \rangle = \ell_{\bar{1}}^{+} \otimes \ell_{1}^{+}, \tag{15}$$

$$\ell_{\bar{1}}^{--} = \langle \cdot \otimes : \otimes 1 \otimes u_1, \boldsymbol{\mathscr{R}} \rangle = \ell_{\bar{1}}^{-} \otimes \ell_{1}^{-}, \tag{16}$$

$$\ell_{\bar{1}}^{-+} = \langle u_1 \otimes 1 \otimes \cdot \otimes :, \boldsymbol{\mathscr{R}}^{-1} \rangle = \langle 1 \otimes u_1 \otimes \cdot \otimes :, \boldsymbol{\mathscr{R}}^{-1} \rangle = \ell_{\bar{1}}^{-} \otimes \ell_{1}^{+}. \tag{17}$$

Choosing $\dot{\mathscr{R}} = \mathscr{R}$ in Equation (3), which gives

$$\boldsymbol{\mathscr{R}}' = \mathscr{R}_{\dot{2}1}^{-1}\mathscr{R}_{\dot{1}\dot{2}}\mathscr{R}_{12}\mathscr{R}_{\dot{1}2} \tag{18}$$

instead of Equation (10), leads to the same set of generators. These generators have matrix comultiplication and, using the decomposition of an invertible matrix in upper and lower triangular parts, it is possible to show that expressions (15)–(17) generate the same Hopf algebra as expressions (14) do. We identify them with those given in [5] and thereby $\boldsymbol{\mathscr{U}}$ with the complexification $\mathscr{U}(\mathbb{C})$ constructed there. The $*$-operation will be investigated in Section 3,

$$
\begin{aligned}
&t^i_j \equiv u^i_j \otimes 1, \qquad \hat{t}^i_j \equiv 1 \otimes u^i_j,\\
&L^{+i}_{j} \equiv \ell^{++i}_{j}, \qquad L^{+\bar{i}}_{j} \equiv \ell^{--i}_{j}, \qquad L^{-i}_{j} = L^{-\bar{i}}_{j} \equiv \ell^{-+i}_{j},
\end{aligned}
\tag{19}
$$

where $u \otimes 1$, $1 \otimes u$ are elements of the dual of $\boldsymbol{\mathscr{U}}$ and t, $\hat{t}$, $L^{\pm}$ were used in [5].

2. The Dual Hopf Algebra of Functions on the Quantum Group

The canonical pairing of the Hopf algebras $\mathscr{U}$ and $\mathscr{A}$, R being the R-matrix

$$R_{12} = \langle u_1 \otimes u_2, \mathscr{R} \rangle, \tag{20}$$

is defined through

$$
\begin{aligned}
&\langle u_1, \ell_2^{+} \rangle = R_{12}, \qquad \langle u_1, \ell_2^{-} \rangle = R_{21}^{-1},\\
&\langle u, 1 \rangle = \varepsilon(u) = 1, \qquad \langle 1, \ell^{\pm} \rangle = \varepsilon(\ell^{\pm}) = 1,\\
&\langle a, gh \rangle = \langle \Delta(a), g \otimes h \rangle, \qquad \langle ab, h \rangle = \langle a \otimes b, \Delta(h) \rangle,\\
&\langle S(a), h \rangle = \langle a, S(h) \rangle.
\end{aligned}
\tag{21}
$$

*$\ell^{++i}_{j} = \ell^{+i}_{k} \otimes \ell^{+k}_{j}$.

If this pairing is nondegenerate, $\mathscr{U}$ and $\mathscr{A}$ are called dually paired [7]. The dual of $\bar{\mathscr{U}}$ is then

$$\bar{\mathscr{A}} = \mathscr{A} \otimes \dot{\mathscr{A}}, \tag{22}$$

its Hopf algebra structure defined in exactly the same way as $\bar{\mathscr{U}}$'s. Their pairing simply factorizes.

Now we proceed to construct $\mathscr{A}$, the dual of $\mathscr{U}$. With the latter adopting unit, counit, and product from $\bar{\mathscr{U}}$, unchanged the former keeps counit, unit, and coproduct from $\bar{\mathscr{A}}$ while a new product and a new antipode are induced by the twisted coproduct and antipode of $\mathscr{U}$, respectively.

For convenience, we introduce the left and right actions of a Hopf algebra on its dual as

$$h \rhd a := a_{(1)}\langle a_{(2)}, h\rangle = \langle \Delta(a), \cdot \otimes h\rangle = \langle a, \cdot h\rangle, \tag{23}$$

$$a \lhd h := \langle a_{(1)}, h\rangle a_{(2)} = \langle \Delta(a), h \otimes \cdot\rangle = \langle a, h \cdot\rangle \tag{24}$$

and register that they fulfill associativity

$$g \rhd (h \rhd a) = (gh) \rhd a, \tag{25}$$

$$(a \lhd g) \lhd h = a \lhd (gh). \tag{26}$$

THEOREM 2. *$\mathscr{A} = \bar{\mathscr{A}}$ as vector space over $\mathbb{C}$ with*

$$\begin{aligned}(a \otimes \dot{a}) \bullet (b \otimes \dot{b}) &= \boldsymbol{m}((a \otimes \dot{a}) \otimes (b \otimes \dot{b})) \\ &= \langle a \otimes \dot{a} \otimes b \otimes \dot{b}, \boldsymbol{\Delta}(\cdot \otimes :)\rangle \\ &= \langle a \otimes \dot{a} \otimes b \otimes \dot{b}, \mathscr{R}^{-1}_{\dot{1}2}\bar{\Delta}(\cdot \otimes :)\mathscr{R}_{\dot{1}2}\rangle \\ &= \langle \dot{a}_{(1)} \otimes b_{(1)}, \mathscr{R}^{-1}\rangle(ab_{(2)} \otimes \dot{a}_{(2)}\dot{b})\langle \dot{a}_{(3)} \otimes b_{(3)}, \mathscr{R}\rangle \\ &= \bar{m}(\mathscr{R}_{\dot{1}2} \rhd (a \otimes \dot{a} \otimes b \otimes \dot{b}) \lhd \mathscr{R}^{-1}_{\dot{1}2}),\end{aligned} \tag{27}$$

$$\boldsymbol{\Delta} = \bar{\Delta}, \tag{28}$$

$$\boldsymbol{\varepsilon} = \bar{\varepsilon}, \tag{29}$$

$$\begin{aligned}\boldsymbol{S}(a \otimes \dot{a}) &= \langle \bar{S}(a \otimes \dot{a}), \mathscr{R}^{-1}_{\dot{1}1}(\cdot \otimes :)\mathscr{R}_{\dot{1}1}\rangle \\ &= \bar{S}(\langle a \otimes \dot{a}, \mathscr{R}_{\dot{1}1}(\cdot \otimes :)\mathscr{R}^{-1}_{\dot{1}1}\rangle) \\ &= \langle \dot{a}_{(1)} \otimes a_{(1)}, \mathscr{R}\rangle \bar{S}(a_{(2)} \otimes \dot{a}_{(2)})\langle \dot{a}_{(3)} \otimes a_{(3)}, \mathscr{R}^{-1}\rangle \\ &= \mathscr{R}_{\dot{1}1} \rhd \bar{S}(a \otimes \dot{a}) \lhd \mathscr{R}^{-1}_{\dot{1}1}\end{aligned} \tag{30}$$

is the dual Hopf algebra of $\mathscr{U}$. The multiplication $\bullet$ was also introduced in [8].

3. $*$-Structures

Now we will construct anti-linear anti-multiplicative involutions for both $\mathscr{A}$ and $\mathscr{U}$ which we denote as $*$. In the following, the matrix $\hat{R}$, as well as its eigenvalues, are supposed to be real. We begin with $\mathscr{A}$.

$$\begin{aligned}(u_2 \otimes u_{\dot{2}})^* \bullet (u_1 \otimes u_{\dot{1}})^* &= [(u_1 \otimes u_{\dot{1}}) \bullet (u_2 \otimes u_{\dot{2}})]^* \\ &= \overline{\langle u_{\dot{1}} \otimes u_2, \mathscr{R}^{-1}\rangle}(u_1 u_2 \otimes u_{\dot{1}} u_{\dot{2}})^* \overline{\langle u_{\dot{1}} \otimes u_2, \mathscr{R}\rangle} \\ &= R^{-1}_{\dot{1}2}(u_1 u_2 \otimes u_{\dot{1}} u_{\dot{2}})^* R_{\dot{1}2}.\end{aligned} \tag{31}$$

We try the ansatz

$$(u_1 \otimes u_{\dot{1}})^* = v_{1\dot{1}} \otimes w_{1\dot{1}} \tag{32}$$

yielding

$$\begin{aligned}(u_2 \otimes u_{\dot{2}})^* \bullet (u_1 \otimes u_{\dot{1}})^* &= (v_{2\dot{2}} \otimes w_{2\dot{2}}) \bullet (v_{1\dot{1}} \otimes w_{1\dot{1}}), \\ &= \langle w_{2\dot{2}(1)} \otimes v_{1\dot{1}(1)}, \mathscr{R}^{-1}\rangle(v_{2\dot{2}} v_{1\dot{1}(2)} \otimes w_{2\dot{2}(2)} w_{1\dot{1}}) \times \\ &\quad \times \langle w_{2\dot{2}(3)} \otimes v_{1\dot{1}(3)}, \mathscr{R}\rangle,\end{aligned} \tag{33}$$

where summation begins with v and ends with w. This suggests

$$\Delta(v_{1\dot{1}}) = v_{1\dot{1}} \otimes v_{1\dot{1}}, \tag{34}$$

$$\Delta(w_{2\dot{2}}) = w_{2\dot{2}} \otimes w_{2\dot{2}}, \tag{35}$$

with

$$\langle w_{2\dot{2}} \otimes v_{1\dot{1}}, \mathscr{R}\rangle = R_{\dot{1}2} \tag{36}$$

and

$$(u_1 u_2 \otimes u_{\dot{1}} u_{\dot{2}})^* = v_{2\dot{2}} v_{1\dot{1}} \otimes w_{2\dot{2}} w_{1\dot{1}}, \tag{37}$$

which are fulfilled by

$$v_{1\dot{1}} = S(1_1 u_{\dot{1}}^{\mathsf{T}}) = S(u_{\dot{1}}^{\mathsf{T}}), \tag{38}$$

$$w_{1\dot{1}} = S(1_{\dot{1}} u_1^{\mathsf{T}}) = S(u_1^{\mathsf{T}}), \tag{39}$$

since

$$\begin{aligned}\langle S(u_{\dot{2}}^{\mathsf{T}}) \otimes S(u_{\dot{1}}^{\mathsf{T}}), \mathscr{R}\rangle &= \langle u_{\dot{2}}^{\mathsf{T}} \otimes u_{\dot{1}}^{\mathsf{T}}, (S \otimes S)(\mathscr{R})\rangle \\ &= \langle u_{\dot{2}}^{\mathsf{T}} \otimes u_{\dot{1}}^{\mathsf{T}}, \mathscr{R}\rangle = R^{\mathsf{T}}_{2\dot{1}} = R_{\dot{1}2}.\end{aligned} \tag{40}$$

The last equality requires symmetry of $\hat{R}$:

$$\hat{R}^{\mathsf{T}} = \hat{R}. \tag{41}$$

Thus, we have (with † being the involution in $\mathscr{U}$)

$$(u_1 \otimes u_{\dot{1}})^* = S(u_{\dot{1}}{}^{\mathsf{T}}) \otimes S(u_1{}^{\mathsf{T}}) = u_{\dot{1}}^{\dagger} \otimes u_1^{\dagger}, \tag{42}$$

which extends, through anti-multiplicativity, to the whole of $\mathscr{A}$. Now we simply induce a $*$-structure on $\mathscr{U}$ by requiring

$$\langle a \otimes \dot{a}, (h \otimes \dot{h})^* \rangle = \overline{\langle S^{-1}((a \otimes \dot{a})^*), h \otimes \dot{h} \rangle}. \tag{43}$$

This turns out to be consistent whenever we have a twisted antipode of the form

$$S' = vSv^{-1}, \tag{44}$$

since the involution we found on the twisted function algebra works on the original one as well ($A^* = A^{\ddagger}$), with ‡ being the involution in the original Hopf algebras, we have

$$\begin{aligned}\langle A, H^* \rangle \\ &= \overline{\langle S'^{-1}(A^{\ddagger}), H \rangle} = \overline{\langle A^{\ddagger}, S'^{-1}(H) \rangle} \\ &= \langle A, S^{-1} \circ {\ddagger} \circ S'^{-1}(H) \rangle = \langle A, (v^{-1}Hv)^{\ddagger} \rangle,\end{aligned} \tag{45}$$

thus in our case, with $(h \otimes \dot{h})^{\ddagger} = \dot{h}^{\dagger} \otimes h^{\dagger}$,

$$\begin{aligned}(h \otimes \dot{h})^* \\ &= (\mathscr{R}_{\dot{1}1}^{-1}(h \otimes \dot{h})\mathscr{R}_{\dot{1}1})^{\ddagger} \\ &= \mathscr{R}_{\dot{1}1}(h \otimes \dot{h})^{\ddagger}\mathscr{R}_{\dot{1}1}^{-1} = \mathscr{R}_{\dot{1}1}(\dot{h}^{\dagger} \otimes h^{\dagger})\mathscr{R}_{\dot{1}1}^{-1},\end{aligned} \tag{46}$$

$$\begin{aligned}\langle u_1{}^{\mathsf{T}} \otimes u_{\dot{1}}{}^{\mathsf{T}}, (h \otimes \dot{h})^* \rangle \\ &= \langle S^{-1}(S(u_{\dot{1}}) \otimes S(u_1)), h \otimes \dot{h} \rangle \\ &= \langle u_{\dot{1}} \otimes u_1, \mathscr{R}_{2\dot{1}}^{-1}(h \otimes \dot{h})\mathscr{R}_{21} \rangle = R_{1\dot{1}}^{-1}\langle u_{\dot{1}} \otimes u_1, h \otimes \dot{h} \rangle R_{1\dot{1}},\end{aligned} \tag{47}$$

$$\begin{aligned}\langle u_1{}^{\mathsf{T}} \otimes u_{\dot{1}}{}^{\mathsf{T}}, (l_2^{+} \otimes l_{\dot{2}}^{+})^* \rangle \\ &= R_{1\dot{1}}^{-1}R_{\dot{1}2}R_{12}R_{1\dot{1}} = R_{12}R_{\dot{1}2} = R_{21}^{\mathsf{T}}R_{2\dot{1}}^{\mathsf{T}} = (R_{2\dot{1}}R_{21})^{\mathsf{T}} \\ &= \langle u_{\dot{1}} \otimes u_1, S(l_{\dot{2}}^{-}) \otimes S(l_2^{-}) \rangle^{\mathsf{T}} \\ &= \langle u_1{}^{\mathsf{T}} \otimes u_{\dot{1}}{}^{\mathsf{T}}, S(l_2^{-\mathsf{T}}) \otimes S(l_{\dot{2}}^{-\mathsf{T}}) \rangle.\end{aligned} \tag{48}$$

In this way, we obtain

$$l_1^{++*} = (l_{\dot{1}}^{+} \otimes l_1^{+})^* = S(l_1^{-\mathsf{T}}) \otimes S(l_{\dot{1}}^{-\mathsf{T}}) = S(l_{\dot{1}}^{--\mathsf{T}}), \tag{49}$$

$$l_1^{--*} = (l_{\dot{1}}^{-} \otimes l_1^{-})^* = S(l_1^{+\mathsf{T}}) \otimes S(l_{\dot{1}}^{+\mathsf{T}}) = S(l_{\dot{1}}^{++\mathsf{T}}), \tag{50}$$

$$l_1^{-+*} = (l_{\dot{1}}^{-} \otimes l_1^{+})^* = S(l_1^{-\mathsf{T}}) \otimes S(l_{\dot{1}}^{+\mathsf{T}}) = S(l_{\dot{1}}^{-+\mathsf{T}}), \tag{51}$$

$$\mathscr{R}^* = (\mathscr{R}_{\dot{2}1}^{-1}\mathscr{R}_{2\dot{1}}^{-1}\mathscr{R}_{12}\mathscr{R}_{\dot{1}\dot{2}})^* = \mathscr{R}^{-1}, \tag{52}$$

$$\mathscr{R}'^* = (\mathscr{R}_{\dot{2}1}^{-1}\mathscr{R}_{\dot{1}2}\mathscr{R}_{12}\mathscr{R}_{\dot{1}\dot{2}})^* = \mathscr{R}'_{2\dot{2}1\dot{1}}. \tag{53}$$

This shows that the generators defined in (15)–(17) have the same reality conditions as those defined in [5] (cf. Equation (19)). Going over to the real representation of the complex quantum group defined through the generators $u_1 \otimes u_{\dot{1}}$, it is possible to define the projection to the sub-quantum groups which corresponds to $\Delta(\mathscr{U})$ in the following way. The elements $A \in \mathscr{A}$ defined by

$$\langle A, H \rangle = 0, \quad \text{where } H \in \Delta(\mathscr{U}), \tag{54}$$

form a bi-ideal. This bi-ideal is generated by the elements

$$P^r_{1\dot{1}}(u_1 \otimes u_{\dot{1}})P^s_{1\dot{1}} \quad \text{for} \quad r \neq s \tag{55}$$

where $P^t_{1\dot{1}}$ are the projectors which belong to the $\hat{R}$ matrix. Dividing out that ideal gives a projection to subgroups.

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